

On the polyhedron of the K -partitioning problem with representative variables

Zacharie ALES^{a,b}, Arnaud KNIPPEL^a, Alexandre PAUCHET^b

^a*LMI INSA Rouen (EA 3226)*

^b*LITIS INSA Rouen (EA 4051)*

Abstract

The K -partitioning problem consists of partitioning the vertices of a graph in K sets so as to minimize a function of the edge weights. We introduce a linear mixed integer formulation with edge variables and representative variables. We consider the corresponding polyhedron and show which inequalities are facet-defining. We study several families of facet-defining inequalities and provide experimental results showing that they improve significantly the linear relaxation of our formulation.

Keywords: combinatorial optimization, polyhedral approach, graph partitioning

1. Introduction

Graph partitioning refers to partitioning the vertices of a graph in several sets, so that a given function of the edge weights is minimized (or maximized). In many papers this function is linear so that minimizing the weight of the edges in the different parts is equivalent to maximizing the total weight of the multicut defined by the node sets. For this reason different names are used in the literature and the most frequent are graph partitioning problem [1, 2, 3] and min-cut problem [4]. When maximizing the cut, with positive weights, the problem is called the max-cut problem [5, 6] and is known to be NP-complete. The problem is sometimes called clique partitioning problem when

*arnaud.knippe@insa-rouen.fr,
+33 2 32 95 66 25,

INSA de Rouen, Avenue de l'Université, 76801, Saint-Etienne du Rouvray

the graph is complete [7, 8], while Chopra and Rao [1] note that the general case can be solved by adding edges to obtain a complete graph. This is the point of view we adopt here for the sake of simplicity. However it is possible to derive specific valid inequalities for sparse graphs, as in [3].

This general problem has many applications (see for example [2]) and many variants, which in most cases are NP-hard [6]. The number of sets in a solution may be specified as a part of the problem definition or not. In this paper we consider the former case that we call the *K-partitioning* problem, where K is the number of sets.

Our motivation comes from a clustering problem for analyzing dialogs in psychology [9]. Dialogs can be encoded using two-dimensional tables (or series of item-sets), along which dialog patterns, representative of human behaviors, are repeated approximately. Partitioning a graph of dialog patterns would enable to group similar instances and therefore to characterize significant behaviors. For this application, instances commonly are a complete graph of 20 to 100 vertices to partition in 6 to 10 sets. Whereas big instances require approximate solutions, we are interested in improving the exact methods based on branch and bound (such as in [10, 11, 12]), by studying more precisely the polyhedral structure of a linear formulation in order to provide better bounds.

A general linear integer formulation using edge variables was proposed in [7], together with several facet-defining inequalities. We call this formulation edge formulation or node-node formulation, and it is often considered stronger than the node-cluster formulation [12], although this may depend on the data sets. However the edge formulation doesn't allow to fix the number of sets easily, as opposite to the node-cluster formulation. Some authors use a formulation with both edge variables and node-cluster variables. A formulation with an exponential number of constraints has been proposed in [13] for a variant of partitioning with a bound on the size of the clusters, applied to sparse graphs. More compact formulations have been proposed based on the linearization of the quadratic formulation [14, 10]. In [14] the weights of the edges are positive and the total edge weight is minimized, while in [10] it is maximized. When considering the triangle inequalities from both formulations [1], graph with arbitrary weights on the edges can be partitioned.

All these formulations have a common drawback: they contain a lot of symmetry and this can considerably slow down methods based on branch and bound or branch and cut. A way to deal with the symmetry is to work on

the branching strategy. Kaibel et al [14] have proposed a general tool, called orbitopal fixing for that purpose. Another way is to break the symmetry directly in the formulation. This approach has already been used in [15] for a vertex coloring problem. More recently a similar idea has been applied to break the symmetry in the node-cluster formulation [12]. In this paper we propose a formulation based on the edge variables of [7, 16] and additional variables that we call representative variables. This allows not only to break the symmetry, but also to fix the number K of sets.

We present in Section 2 our formulation. We study in Section 3 the dimension of the polyhedron $P_{n,K}$ associated to our formulation. We characterize in Section 4 all the trivial facet-defining inequalities. In Sections 5, 6 and 7, we respectively study the so-called 2-chorded cycle inequalities, the 2-partition inequalities and the general clique inequalities and we determine cases where they define facets of $P_{n,K}$. In Section 8 we strengthen the triangle inequalities from our formulation that do not define facets and we show that most of the time the strengthened triangle inequalities are facet-defining. In Section 9 we study a new family of inequalities called *paw inequalities* and identify when they are facet-defining. In the last section we illustrate the improvement on the linear relaxation value of our formulation for the facet-defining inequalities of the previous sections, for complete graphs with different kind of weights.

2. A mixed integer linear formulation

Let $V = \{1, \dots, n\}$ be a set of indexed vertices and $G = (V, E)$ the complete graph induced by V . A K -partition π is a collection of K non-empty subsets C_1, C_2, \dots, C_K , called clusters, such that $\forall i \neq j, C_i \cap C_j = \emptyset$ and $\bigcup_{i=1}^K C_i = V$.

To each K -partition π , we associate a characteristic vector $x^\pi \in \{0, 1\}^{|E|+|V|}$ such that:

- for each edge $uv \in E$, $x_{u,v}^\pi$ (equivalent to $x_{v,u}^\pi$) is equal to 1 if $u, v \in C_i$ for some i in $\{1, 2, \dots, K\}$ and 0 otherwise;
- for each vertex $u \in V$, $x_u^\pi = 1$ if u is the vertex with the smallest index of its cluster (in that case u is said to be the *representative* of its cluster) and 0 otherwise.

An edge $uv \in E$ is said to be *activated* for a given partition π if $x_{u,v}^\pi = 1$. In this context, u is said to be *linked* to v and vice versa. A vertex i is said to be *lower* than another vertex j (noted $i < j$) if index i is lower than index j .

Let $d_{i,j}$ denote the cost of edge $ij \in E$. We consider the following formulation for the K -partitioning problem:

$$(P_1) \left\{ \begin{array}{ll} \min \sum_{ij \in E} d_{i,j} x_{i,j} & \\ x_{i,k} + x_{j,k} - x_{i,j} \leq 1 & \forall i, j, k \in V, i \neq k, j \neq k, i < j \quad (1) \\ x_j + x_{i,j} \leq 1 & \forall i, j \in V, i < j \quad (2) \\ x_j + \sum_{i=1}^{j-1} x_{i,j} \geq 1 & \forall j \in V \quad (3) \\ \sum_{i=1}^n x_i = K & \quad (4) \\ x_{i,j} \in \{0, 1\} & ij \in E \\ x_i \in [0, 1] & i \in V \end{array} \right.$$

Constraints (1), called *triangle inequalities*, ensure that if two incident edges ij and jk are activated then ik is also activated. Note that there are $\frac{n(n-1)(n-2)}{2}$ such constraints (three for each triangle a, b, c of $G = (V, E)$). Constraints (2), called *upper representative inequalities*, ensure that every cluster contains no more than one representative. If j is a representative then it is not linked to any lower vertex i . If j is linked to such a lower vertex then it is not a representative. Constraints (3), called *lower representative inequalities*, guarantee that a cluster contains at least one representative. Indeed, on the one hand if j is not a representative then it is linked to at least one lower vertex, on the other hand if j is not linked to any of these vertices then it is a representative. Finally, constraint (4) ensures that the number of clusters is equal to K .

Note that in the above mixed linear program the representative variables can be relaxed as fixing all edge variables to 0 or 1 forces the representative variables to be in $\{0, 1\}$. Hence, we only have $|E|$ binary variables.

As the polyhedron associated to the above formulation is not full-dimensional, we fix x_1 to 1 (since vertex 1 is always a representative) and substitute x_2 by $1 - x_{1,2}$ and x_3 by $K - 2 + x_{1,2} - \sum_{i=4}^n x_i$. Equation (4) can now be removed since the number of clusters in a solution is ensured to be K by the

expression substituted to x_3 . These modifications lead to the following linear mixed integer program:

$$(P_2) \left\{ \begin{array}{ll} \min \sum_{ij \in E} w_{i,j} x_{i,j} & \\ x_{i,j} + x_{j,k} - x_{i,k} \leq 1 & \forall i, j, k \in V, i \neq j, i \neq k, j < k \quad (1) \\ x_j + x_{i,j} \leq 1 & \forall i \in V, j \in V - \{1, 2, 3\}, i < j \quad (2) \\ \sum_{j=4}^n x_j - x_{1,2} - x_{i,3} \geq K - 3 & \forall i \in \{1, 2\} \quad (2') \\ x_j + \sum_{i=1}^{j-1} x_{i,j} \geq 1 & \forall j \in V \quad (3) \\ \sum_{i=4}^n x_i - x_{1,2} - x_{1,3} - x_{2,3} \leq K - 3 & \quad (3') \\ \sum_{i=4}^n x_i - x_{1,2} \geq K - 3 & \quad (5) \\ \sum_{i=4}^n x_i - x_{1,2} \leq K - 2 & \quad (6) \\ x_{i,j} \in \{0, 1\} & ij \in E \\ x_i \in [0, 1] & i \in V \end{array} \right. .$$

As a result, the characteristic vector of a partition π no longer contains the components x_1 , x_2 and x_3 . The vector x^π now contains the $n - 3$ remaining representative components followed by the $|E|$ edges components:

$$(x^\pi)^T = (x_4, \dots, x_n, x_{1,2}, \dots, x_{1,n}, x_{2,3}, \dots, x_{n-1,n}).$$

However, for a given vector $\alpha \in \mathbb{R}^{|E|+|V|-3}$ the three coefficients related to x_1 , x_2 and x_3 (α_1 , α_2 and α_3) may appear in subsequent proofs to assist the understanding. In that case, they are equal to zero.

Let $P_{n,K}$ be the convex hull of all integer points which are feasible for (P_2) :

$$P_{n,K} = \text{conv}\{x \in \{0, 1\}^{|E|+|V|-3} \mid x \text{ satisfies } (1), (2), (2'), (3), (3'), (5), (6)\}$$

To simplify the notations, a singleton $\{s\}$ may be denoted by s . Likewise for a given vector $\alpha \in \mathbb{R}^{|E|+|V|-3}$ and a subset E_1 of E , the term $\alpha(E_1)$ is

used to denote the sum of the α components in E_1 ($\sum_{e \in E_1} \alpha_e$). Finally, if we consider two subsets of V , V_1 and V_2 , the sum of the α inter-set components $\sum_{i \in V_1} \sum_{j \in V_2} \alpha_{i,j}$ and the sum of the α intra-set components $\sum_{i,j \in V_1, i < j} \alpha_{i,j}$ are respectively denoted as $\alpha(V_1, V_2)$ and $\alpha(V_1)$.

The truth function $\mathbb{1}$ of a boolean expression e (noted $\mathbb{1}\{e\}$) is equal to 1 if e is true and 0 otherwise.

Given two disjoint clusters C_1, C_2 and a set of vertices $R \subset C_1 \cup C_2$ we define the following transformation: $\mathcal{T} : \{C_1, C_2, R\} \mapsto \{C'_1, C'_2\}$, with $C'_1 = (C_1 \setminus R) \cup (R \setminus C_1)$ and $C'_2 = (C_2 \setminus R) \cup (R \setminus C_2)$. The corresponding transformation is presented in Figure 1. This operator will be used in the following to highlight relations between the coefficients of hyperplanes of $\mathbb{R}^{|E|+|V|-3}$.

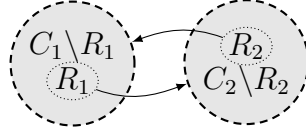


Figure 1: Representation of $\mathcal{T}(C_1, C_2, R)$ with $R_1 = R \cap C_1$ and $R_2 = R \cap C_2$.

3. Dimension of $P_{n,K}$

Let $\pi = \{C_1, C_2, \dots, C_K\}$ be a K -partition. For all $i \in \{1, \dots, K\}$, the representative vertices of clusters C_i is referred to as r_i . The second lowest vertex of this cluster is denoted by $r'_i = \min\{j \in C_i \setminus \{r_i\}\}$.

To prove that $P_{n,K}$ is full-dimensional if and only if $K \in \{3, 4, \dots, n-2\}$, we first assume that it is included in a hyperplane $H = \{x \in \mathbb{R}^{|E|+|V|-3} \mid \alpha^T x = \alpha_0\}$. We then prove that every α coefficient is equal to zero. To this end, we rely on four lemmas (summed up in table 1).

Lemma 3.1. *Consider the four following K -partitions :*

- (i) $\pi = \{C_1, C_2, C_3, \dots, C_K\}$ with $c_1 \in C_1 \setminus \{r_1\}$ and $c_2 \in C_2 \setminus \{r_2\}$;
- (ii) $\pi^1 = \{C_1^{(1)}, C_2^{(1)}, C_3, \dots, C_K\}$ with $\{C_1^{(1)}, C_2^{(1)}\} = \mathcal{T}(C_1, C_2, \{c_1\})$;
- (iii) $\pi^2 = \{C_1^{(2)}, C_2^{(2)}, C_3, \dots, C_K\}$ with $\{C_1^{(2)}, C_2^{(2)}\} = \mathcal{T}(C_1, C_2, \{c_2\})$;
- (iv) $\pi^3 = \{C_1^{(3)}, C_2^{(3)}, C_3, \dots, C_K\}$ with $\{C_1^{(3)}, C_2^{(3)}\} = \mathcal{T}(C_1, C_2, \{c_1, c_2\})$.

If $x^\pi, x^{\pi^1}, x^{\pi^2}, x^{\pi^3}$ satisfy $\alpha^T x = \alpha_0$ then $\alpha_{c_1, c_2} = 0$.

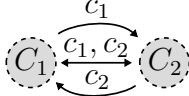
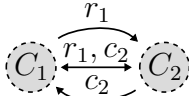
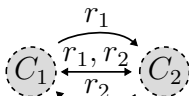
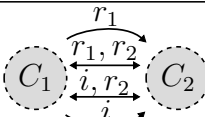
Lemma	Conditions	Valid transformations	Result
3.1	$c_1 \in C_1 \setminus \{r_1\},$ $c_2 \in C_2 \setminus \{r_2\}$		$\alpha_{c_1, c_2} = 0$
3.2	$r_1 < r_2$ $c_2 \in C_2 \setminus \{r_2\}$		$2\alpha_{r_1, c_2} + \alpha_{r'_1} = \alpha_{\min(r'_1, c_2)}$
3.3	$r_1 < r_2$ $r_2 < r'_1$ $ C_2 \geq 2$		$2\alpha_{r_1, r_2} + \alpha_{r'_1} + \alpha_{r'_2} = 2\alpha_{r_2}$
3.4	$i \in C_1 \setminus \{r_1\}$ $i < r_2$		$\alpha_{r_1, r_2} = \alpha_{r_2, i}$

Table 1: Summary of the four lemmas. Given a K -partition $\pi = \{C_1, \dots, C_K\}$ whose vector x^π is included in a hyperplane $H = \{x \in \mathbb{R}^{|E|+|V|-3} \mid \alpha^T x = \alpha_0\}$, each arrow of the third column corresponds to a transformation on C_1 and C_2 which has to give a K -partition included in H .

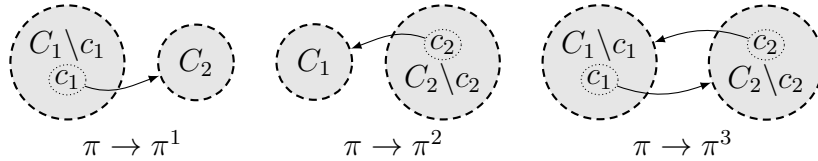


Figure 2: Representation of the transformations which lead from π to π^1 , π^2 and π^3 .

Proof. Let m_2 denote $\min(r_2, c_1)$. Using the fact that $\alpha^T x^\pi$ and $\alpha^T x^{\pi^1}$ are both equal to α_0 , we obtain

$$\sum_{i=1}^K (\alpha(C_i) + \alpha_{r_i}) = \sum_{i=3}^K (\alpha(C_i) + \alpha_{r_i}) + \alpha(C_1 \setminus \{c_1\}) + \alpha(C_2 \cup \{c_1\}) + \alpha_{r_1} + \alpha_{m_2}. \quad (7)$$

This can be simplified and reformulated as

$$\alpha(\{c_1\}, C_1) - \alpha(\{c_1\}, C_2 \setminus \{c_2\}) = \alpha_{c_1, c_2} + \alpha_{m_2} - \alpha_{r_2}. \quad (8)$$

Let m_1 correspond to $\min(r_1, c_2)$. Similarly we obtain from π^2

$$\alpha(\{c_2\}, C_2) - \alpha(\{c_2\}, C_1 \setminus \{c_1\}) = \alpha_{c_1, c_2} + \alpha_{m_1} - \alpha_{r_1}. \quad (9)$$

Finally, π^3 leads to

$$\alpha(\{c_1\}, C_1) + \alpha(\{c_2\}, C_2) + \alpha_{r_1} + \alpha_{r_2} = \alpha(\{c_2\}, C_1 \setminus \{c_1\}) + \alpha(\{c_1\}, C_2 \setminus \{c_2\}) + \alpha_{m_1} + \alpha_{m_2} \quad (10)$$

From (8), (9) and (10) we obtain $\alpha_{c_1 c_2} = 0$. \square

Lemma 3.2. *Consider the four following K -partitions :*

- (i) $\pi = \{C_1, C_2, C_3, \dots, C_K\}$ with $r_1 < r_2$, $c_2 \in C_2 \setminus \{r_2\}$ and $|C_1| \geq 2$;
- (ii) $\pi^1 = \{C_1^{(1)}, C_2^{(1)}, C_3, \dots, C_K\}$ with $\{C_1^{(1)}, C_2^{(1)}\} = \mathcal{T}(C_1, C_2, \{r_1\})$;
- (iii) $\pi^2 = \{C_1^{(2)}, C_2^{(2)}, C_3, \dots, C_K\}$ with $\{C_1^{(2)}, C_2^{(2)}\} = \mathcal{T}(C_1, C_2, \{c_2\})$;
- (iv) $\pi^3 = \{C_1^{(3)}, C_2^{(3)}, C_3, \dots, C_K\}$ with $\{C_1^{(3)}, C_2^{(3)}\} = \mathcal{T}(C_1, C_2, \{r_1, c_2\})$.

If $x^\pi, x^{\pi^1}, x^{\pi^2}, x^{\pi^3}$ satisfy $\alpha^T x = \alpha_0$ then $2\alpha_{r_1, c_2} + \alpha_{r'_1} = \alpha_{\min(r'_1, c_2)}$.

Proof. The transformation which leads from π to π^1 is represented in Figure 3. Since $\alpha^T x^\pi = \alpha^T x^{\pi^1}$ we deduce:

$$\alpha(\{r_1\}, C_1) + \alpha_{r_2} = \alpha(\{r_1\}, C_2) + \alpha_{r'_1}. \quad (11)$$

In π^1 , r_2 is not a representative vertex since r_1 - which is lower - is in its cluster. As a result α_{r_2} only appears in the left part of the equation. At the opposite, r'_1 is not a representative in π , but becomes one when r_1 is moved to C_2 . Therefore, $\alpha_{r'_1}$ appears in the right part of the equation $\alpha^T x^\pi = \alpha^T x^{\pi^1}$.

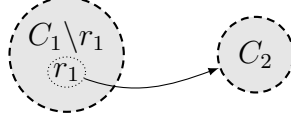


Figure 3: Representation of the transformation which leads from π to π^1 when $c_1 = r_1$.

We deduce from $\alpha^T x^\pi = \alpha^T x^{\pi^2}$:

$$\alpha(\{c_2\}, C_2) = \alpha(\{c_2\}, C_1). \quad (12)$$

Furthermore, the fact that $\alpha^T x^\pi$ is equal to $\alpha^T x^{\pi^3}$ gives

$$\begin{aligned} \alpha(\{r_1\}, C_1) + \alpha(\{c_2\}, C_2) + \alpha_{r_2} = \\ \alpha(\{r_1\}, C_2 \setminus \{c_2\}) + \alpha(\{c_2\}, C_1 \setminus r_1) + \alpha_{\min(r'_1, c_2)}. \end{aligned} \quad (13)$$

Finally, this result can be simplified to $\alpha_{r_1, c_2} + \alpha_{r_2} = \alpha_{\min(r'_1, c_2)}$ with equations (11) and (12). \square

Lemma 3.3. *Consider the four following K -partitions :*

- (i) $\pi = \{C_1, C_2, C_3, \dots, C_K\}$ with $r_1 < r_2 < r'_1$ and $|C_2| \geq 2$;
- (ii) $\pi^1 = \{C_1^{(1)}, C_2^{(1)}, C_3, \dots, C_K\}$ with $\{C_1^{(1)}, C_2^{(1)}\} = \mathcal{T}(C_1, C_2, \{r_1\})$;
- (iii) $\pi^2 = \{C_1^{(2)}, C_2^{(2)}, C_3, \dots, C_K\}$ with $\{C_1^{(2)}, C_2^{(2)}\} = \mathcal{T}(C_1, C_2, \{r_2\})$;
- (iv) $\pi^3 = \{C_1^{(3)}, C_2^{(3)}, C_3, \dots, C_K\}$ with $\{C_1^{(3)}, C_2^{(3)}\} = \mathcal{T}(C_1, C_2, \{r_1, r_2\})$.

If $x^\pi, x^{\pi^1}, x^{\pi^2}, x^{\pi^3}$ satisfy $\alpha^T x = \alpha_0$ then $2\alpha_{r_1, r_2} + \alpha_{r'_1} + \alpha_{r'_2} = 2\alpha_{r_2}$.

Proof. The fact that $\alpha^T x^\pi$ is equal to $\alpha^T x^{\pi^1}$ leads again to equation (11).

From $\alpha^T x^\pi = \alpha^T x^{\pi^2}$ we deduce:

$$\alpha(\{r_2\}, C_2) + r_2 = \alpha(\{r_2\}, C_1) + r'_2. \quad (14)$$

The vertex r_2 is no more a representative in π^1 since it is in the same cluster than r_1 . r'_2 becomes a representative since r_2 is not in its cluster anymore.

Finally, $\alpha^T x^\pi = \alpha^T x^{\pi^3}$ gives

$$\alpha(\{r_1\}, C_1) + \alpha(\{r_2\}, C_2) = \alpha(\{r_1\}, C_2 \setminus \{r_2\}) + \alpha(\{r_2\}, C_1 \setminus \{r_1\}). \quad (15)$$

This last relation can be simplified to $2\alpha_{r_1, r_2} + \alpha_{r'_1} + \alpha_{r'_2} = 2\alpha_{r_2}$ via equations (11) and (14). \square

Lemma 3.4. Consider the five following K -partitions :

- (i) $\pi = \{C_1, C_2, C_3, \dots, C_K\}$ with $i \in C_1 \setminus \{r_1\}$, $i < r_2$;
- (ii) $\pi^1 = \{C_1^{(1)}, C_2^{(1)}, C_3, \dots, C_K\}$ with $\{C_1^{(1)}, C_2^{(1)}\} = \mathcal{T}(C_1, C_2, \{r_1\})$;
- (iii) $\pi^2 = \{C_1^{(2)}, C_2^{(2)}, C_3, \dots, C_K\}$ with $\{C_1^{(2)}, C_2^{(2)}\} = \mathcal{T}(C_1, C_2, \{i\})$;
- (iv) $\pi^3 = \{C_1^{(3)}, C_2^{(3)}, C_3, \dots, C_K\}$ with $\{C_1^{(3)}, C_2^{(3)}\} = \mathcal{T}(C_1, C_2, \{r_1, r_2\})$;
- (v) $\pi^4 = \{C_1^{(4)}, C_2^{(4)}, C_3, \dots, C_K\}$ with $\{C_1^{(4)}, C_2^{(4)}\} = \mathcal{T}(C_1, C_2, \{i, r_2\})$.

If $x^\pi, x^{\pi^1}, x^{\pi^2}, x^{\pi^3}, x^{\pi^4}$ satisfy $\alpha^T x = \alpha_0$ then $\alpha_{r_1, r_2} = \alpha_{r_2, i}$.

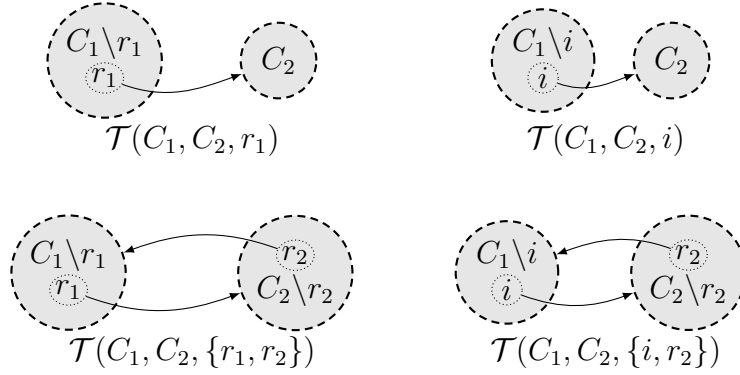


Figure 4: Representation of the transformations which lead from π to $\pi_{r_1}^1$, π_i^1 , π^3 and π^4 .

Proof. As shown in the previous proofs, we respectively obtain from $\alpha^T x^\pi = \alpha^T x^{\pi_{r_1}^1}$ and $\alpha^T x^\pi = \alpha^T x^{\pi_i^1}$ equations (11) and (8) (with $c_2 = r_2$ in this last case).

From $\alpha^T x^\pi = \alpha^T x^{\pi^3}$ (represented Figure 4) we deduce:

$$\alpha(\{r_1\}, C_1) + \alpha(\{r_2\}, C_2) + \alpha_{r_2} = \alpha(\{r_1\}, C_2 \setminus \{r_2\}) + \alpha(\{r_2\}, C_1 \setminus \{r_1\}) + \alpha_{r_1'}. \quad (16)$$

The set $C_1^{(3)}$ is equal to $\{C_1 \setminus \{r_1\}\} \cup \{r_2\}$. Since r_1' is lower than r_2 , it is $C_1^{(3)}$ representative.

The equality $\alpha^T x^\pi = \alpha^T x^{\pi^4}$ shows that

$$\alpha(\{i\}, C_1) + \alpha(\{r_2\}, C_2) + \alpha_{r_2} = \alpha(\{i\}, C_2 \setminus \{r_2\}) + \alpha(r_2, C_1 \setminus \{i\}) + \alpha_i. \quad (17)$$

From equations (11) and (16) we get:

$$\alpha(\{r_2\}, C_2) + \alpha_{r_1, r_2} = \alpha(\{r_2\}, C_1 \setminus \{r_1\}), \quad (18)$$

and from equations (8) and (17) we obtain:

$$\alpha(\{r_2\}, C_2) + \alpha_{i, r_2} = \alpha(\{r_2\}, C_1 \setminus \{i\}). \quad (19)$$

Finally, the two last equations yield the expected result. \square

Theorem 3.5. *Depending on K , the dimension of $P_{n,K}$ is:*

- (i) $\dim(P_{n,2}) = |E| + n - 4$;
- (ii) $\dim(P_{n,K}) = |E| + n - 3$, for $K \in \{3, 4, \dots, n-2\}$ (i.e.: it is full dimensional);
- (iii) $\dim(P_{n,n-1}) = |E| - 1$.

Proof. $P_{n,n-1}$ contains exactly $|E|$ integer solutions which corresponds to all the partitions with exactly one edge activated. These solutions are independent and thus $\dim(P_{n,n-1}) = |E| - 1$.

We now consider $K < n - 1$. Assume that $P_{n,K}$ is included in $H = \{x \in \mathbb{R}^{|E|+|V|-3} \mid \alpha^T x = \alpha_0\}$. We prove that H is unique if $K = 2$ and that all its coefficients are equal to 0 if $K \in \{3, 4, \dots, n-2\}$. Unless stated otherwise, the K -partitions considered throughout the remainder of this proof only require that two of their clusters contain more than one element. They are thus feasible since K is assumed to be lower than $n - 1$.

We first apply Lemma 3.1 with $r_1 = 1$, $r_2 = 2$ and $c_1, c_2 \in \{3, 4, \dots, n\}$ to obtain that $\alpha_{c_1, c_2} = 0$. Similarly, if $r_2 = 3$ and $c_1 = 2$, we get that $\alpha_{2, c_2} = 0$ for all $c_2 \geq 4$.

Furthermore, if $\{1, 3\} \subset C_1$, $\{2\} \subset C_2$ and $c_2 \in \{4, 5, \dots, n\}$, Lemma 3.2 states: $\alpha_{1, c_2} = 0$. Up to this level, we know that $\alpha_{i, j} = 0$ for all $ij \in E \setminus \{12, 13, 23\}$.

Lemma 3.4 for $r_1 = 1$, $i = 2$ and $r_2 = 3$ shows that $\alpha_{1,3}$ and $\alpha_{2,3}$ are equal to a value that will be referred to as β .

Moreover, if $r_1 = 1$, $r_2 = 2$, $c_2 = 3$ and $c_1 \in \{4, 5, \dots, n\}$, Lemma 3.2 can be used to highlight that $2\beta + \alpha_{c_1} = \alpha_3$. As previously stated, the variable

x_3 has been removed from the formulation and its corresponding coefficient α_3 is equal to 0. We obtain that for all $s \in \{4, 5, \dots, n\}$, $\alpha_s = -2\beta$.

We now use Lemma 3.3 with $\{1, 3\} \subset C_1$, $r_2 = 2$ and $r'_2 \in \{4, 5, \dots, n\}$ to get: $2\alpha_{1,2} + \alpha_{r'_2} = 0$. We then conclude that $\alpha_{1,2} = \beta$.

At this point, we know that $\alpha_{1,2} = \alpha_{1,3} = \alpha_{2,3} = \beta$, that for all $s \in \{4, 5, \dots, n\}$ $\alpha_s = -2\beta$ and that all the other coefficients are equal to 0. Thus, the only hyperplane which can contain $P_{n,K}$ is

$$\beta(x_{1,2} + x_{1,3} + x_{2,3} - 2 \sum_{s=4}^n x_s) = \alpha_0. \quad (20)$$

Each cluster which does not contain the vertices 1, 2 or 3 has exactly one vertex whose representative variable has value one. Moreover, if $K = 2$, the three first vertices can either be linked or scattered in the two clusters. In both cases, it can be checked that equation (20) is always satisfied if $\alpha_0 = \beta$. As a consequence, exactly one hyperplane contains $P_{n,2}$. Its dimension is thus $|E| + n - 4$.

If $K \in \{3, 4, \dots, n-2\}$ Lemma 3.3 is used with $r_1 = 1$, $r_2 = 2$ and $3 \in C_3$ to show that $2\beta - 4\beta = 0$. β is, therefore, equal to 0. In the general case, we conclude that there is no hyperplane which contains $P_{n,K}$. Therefore, its dimension is maximal. □

Thereafter, we study facet-defining inequalities for $P_{n,K}$ when it is full-dimensional (*i.e.*: $K \in \{3, \dots, n-2\}$). For each studied face $F = \{x \in P_{n,K} | \omega^T x = \omega_0\}$, we consider a facet-defining inequality $\alpha^T x \leq \alpha_0$ such that $F \subseteq \{x \in P_{n,K} | \alpha^T x = \alpha_0\}$. We then prove that F is facet-defining by highlighting, with reference to Theorem 3.6 in Section I.4.3 of [17], that (α, α_0) is proportional to (ω, ω_0) .

4. Trivial facets

In this section, we show which of the inequalities from the integer formulation are facet-defining. We restrict our study to the general cases where $P_{n,K}$ is full-dimensional (*i.e.*: $K \in \{3, 4, \dots, n-2\}$).

4.1. Edge bound inequalities

Remark The inequalities $x_{u,v} \leq 1$ for all $uv \in E$ are not facet-defining since they are induced by the two following inequalities: $x_{u,v} + x_{u,i} - x_{v,i} \leq 1$ and $x_{u,v} + x_{v,i} - x_{u,i} \leq 1$ for all $i \in V \setminus \{u, v\}$.

Theorem 4.1. *If $P_{n,K}$ is full-dimensional, the inequalities $x_{u,v} \geq 0$ are facet-defining if and only if $uv \notin \{12, 13, 23\}$.*

Proof. We first show that if $uv \in \{12, 13, 23\}$ then $x_{u,v} \geq 0$ is not facet-defining. If $x_{1,2} = 0$, vertex 3 is either linked to 1, 2 or none of them. The sum of representatives $\sum_{i=4}^n x_i$ is equal to $K - 2$ for the two first cases and to $K - 3$ for the last one. We then deduce that the face of $P_{n,K}$ defined by $x_{1,2} \geq 0$ is not a facet since it is also included in the hyperplane defined by $\sum_{i=4}^n x_i = x_{1,3} + x_{2,3} + K - 3$. Symmetric reasonings yield similar results for $x_{1,3}$ and $x_{2,3}$.

We now consider an edge $uv \in E$ such that $v \geq 4$ and we show: $F_{u,v} = \{x^\pi \in P_{n,K} \mid x_{u,v}^\pi = 0\}$ is a facet of $P_{n,K}$. To that end, we consider a hyperplane $H = \{x \in \mathbb{R}^{|E|+|V|-3} \mid \alpha^T x = \alpha_0\}$ which includes $F_{u,v}$ and we prove that all its coefficients with the exception of $\alpha_{u,v}$ are equal to zero. $H = \{x \in \mathbb{R}^{|E|+|V|-3} \mid \alpha^T x = \alpha_0\}$ and that - except $\alpha_{u,v}$ - all the coefficients of α are equal to zero.

The transformations used in the first part of Theorem 3.5 can be used again here by setting $C_3 = \{v\}$ in order to ensure: $x_{u,v} = 0$. We thus obtain:

- $\alpha_{i,j} = 0 \quad \forall ij \in E \setminus \{12, 13, 23\}, i \neq v, j \neq v;$
- $\alpha_{1,2} = \alpha_{1,3} = \alpha_{2,3} \stackrel{\text{def}}{=} \beta;$
- $\alpha_i = -2\beta \quad \forall i \neq v.$

Then we apply Lemma 3.1 with r_1 and r_2 in $\{1, 2, 3\} \setminus \{u\}$, $v \in C_1, C_3 = \{u\}$ and $i \in C_2$ with $i \in \{4, 5, \dots, n\} \setminus \{u, v\}$ to deduce: $\alpha_{i,v} = 0$. It remains to prove that $\alpha_{a,v} = \alpha_v = \beta = 0$ for all $a \in \{1, 2, 3\} \setminus \{u\}$.

Let $b \in \{1, 2, 3\} \setminus \{a, u\}$. The transformations $\mathcal{T}(\{a, b, u\}, v, a)$ and $\mathcal{T}(\{a, v\}, u, a)$ (Figure 5 and 6) lead to $\beta = 0$ and $\alpha_{a,v} = \alpha_v$. Eventually, the transformation $\mathcal{T}(\{a, b, v\}, u, \{a, b\})$ (Figure 7) gives $2\alpha_{a,v} = \alpha_v$ which leads to $\alpha_{a,v} = \alpha_v = 0$. \square

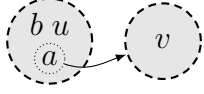


Figure 5:
 $\mathcal{T}(\{a, b, u\}, v, a)$

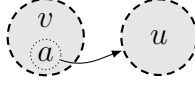


Figure 6:
 $\mathcal{T}(\{a, v\}, u, a)$

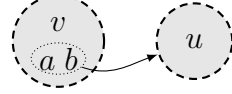


Figure 7:
 $\mathcal{T}(\{a, b, v\}, u, a)$

4.2. Representative bound inequalities

Remark The inequalities $x_v \leq 1$ for all $v \in \{4, 5, \dots, n\}$ are not facet-defining since the face induced by $x_v = 1$ is contained in the hyperplanes $\{x \in \mathbb{R}^{|E|+|V|-3} \mid x_{u,v} = 0\}$ for all $u \in \{1, 2, \dots, v-1\}$. Indeed, if v is the representative of a cluster C , it must be the lowest vertex of C . The dimension of the face induced by $x_v \leq 1$ is thus lower than or equal to $\dim(P_{n,K}) - v + 1$.

Remark For the same reason, the inequality $\sum_{i=4}^n x_i - x_{1,2} \geq K - 3$ which corresponds to $x_3 \leq 1$ is not facet-defining. Neither is $x_3 \geq 0$ since in that case $x_{1,3} + x_{2,3} - x_{1,2} = 1$ (*i.e.*: vertex 3 is not a representative so it is either with 1, 2 or both).

Theorem 4.2. *If $P_{n,K}$ is full-dimensional, the inequalities $x_v \geq 0$ for all $v \in \{4, 5, \dots, n\}$, are facet-defining if and only if $K \neq n - 2$.*

Proof. We first prove that, if $K = n - 2$, the inequalities $x_v \geq 0$ are not facet-defining. In that case, only two vertices are not representative and the face induced by $x_v \geq 0$ is thus included in the hyperplane defined by $\sum_{i=4}^n x_i = x_{1,2} + x_{1,3} + x_{2,3} + K - 3$. Indeed, the sum of the representative variables $\sum_{i=4}^n x_v$ can, vary from $K - 3$ (if each 1, 2 and 3 is in a cluster reduced to one vertex) to $K - 1$ (if the three first vertices are in the same cluster).

Given a valid partition $\pi = \{C_1, C_2, \dots, C_K\}$ we know since $K \leq n - 3$ that at least three vertices of V are not representative of their cluster. As a consequence, the partitions considered in the first case of the proof of Theorem 3.5 are still valid if we add vertex v to C_1 or C_2 . Moreover, it is easy to check that we can always add v to C_1 or C_2 in such a way that v is never a representative of its cluster (it is always in a cluster with at least one vertex among 1, 2 and 3). Therefore, by following the same reasoning we obtain that if the face of $P_{n,K}$ defined by $x_v \geq 0$ is included in a hyperplane $H = \{x \in \mathbb{R}^{|E|+|V|-3} \mid \alpha^T x = \alpha_0\}$ its only non-zero coefficient is α_v . Thus, the only hyperplane which contains the face is $x_v = 0$.

□

4.3. Upper representative inequalities

Theorem 4.3. *If $P_{n,K}$ is full-dimensional, the inequalities $x_{u,v} + x_v \leq 1$ for all $v \geq 4$ and all $u < v$ are facet-defining if and only if $n \geq 6$ or $\{u, v\} \neq \{4, 5\}$.*

Proof. $P_{n,K}$ is full-dimensional if and only if $K \in \{3, 4, \dots, n-2\}$. Therefore, n is greater than four and if it is equal to five, K must be equal to three. As a result, if $n = 5$, $u = 4$ and $v = 5$, every K -partition which satisfies $x_{4,5} + x_5 = 1$ is contained in the hyperplane induced by $2(x_4 + x_5) + \sum_{i \leq 3} x_{i,5} = x_{1,2} + x_{1,3} + x_{2,3} + 1$.

In other cases, we assume that the face $F_{u,v} = \{x \in P_{n,K} \mid x_{u,v} + x_v = 1\}$ is included in a hyperplane induced by: $\alpha^T x = \alpha_0$. To start with, we deduce, similarly to the proof of Theorem 4.1, that $\alpha_{i,j} = 0 \ \forall ij \in E \setminus \{12, 13, 23\}$ with $i \neq v$ and $j \neq v$; $\alpha_{1,2} = \alpha_{1,3} = \alpha_{2,3} \stackrel{\text{def}}{=} \beta$; $\alpha_i = -2\beta \ \forall i \neq v$. The transformation $\mathcal{T}(\{u, v, i\}, \{j\}, \{i\})$ with $i \in V \setminus \{u, v\}$ and $j \in \{1, 2, 3\} \setminus \{i, u\}$ gives $\alpha_{i,v} = 0$.

We now consider a partition $\pi = \{C_1, C_2, \dots, C_K\}$ such that:

- $C_1 = \{a, b\}$, with a and b two distinct vertices lower than 4;
- $C_2 = i$, with $i \geq 4$ and different from u and v (which is always possible whether $n \geq 6$ or $\{u, v\} \neq \{4, 5\}$);
- $\{u, v\} \subset C_3$.

The transformation $\mathcal{T}(\{a, b\}, i, a)$ (Figure 8) shows that β is equal to zero. Eventually, the transformation $\mathcal{T}(\{a, u\}, v, u)$ leads to $\alpha_v = \alpha_{u,v}$ which concludes this proof.

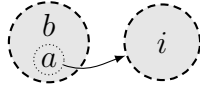


Figure 8: $\mathcal{T}(\{a, b\}, i, a)$

□

Theorem 4.4. *If $P_{n,K}$ is full-dimensional the inequalities $x_{1,2} + x_{a,3} - \sum_{i=4}^n x_i \leq 3 - K$ for $a \in \{1, 2\}$ - which correspond to $x_{a,3} + x_3 \leq 1$ - are facet-defining.*

Proof. We assume that the face $F_u = \{x \in P_{n,K} | x_{1,2} - \sum_{i=4}^n x_i + x_{a,3} = 3, K\}$ of $P_{n,K}$ is included in a hyperplane $H = \{x \in \mathbb{R}^{|E|+|V|-3} | \alpha^T x = \alpha_0\}$ and we show that $\alpha^T x$ is equal to $\alpha_{1,2}x_{1,2} + \alpha_{a,3}x_{a,3} - \sum_{i=4}^n \alpha_i x_i$. Let b be whichever vertex, 1 or 2, is different from a . We use Lemma 3.1 with $r_1 = a$, $r_2 = 3$ and $b \in C_3$ to show $\alpha_{i,j} = 0 \ \forall i, j \geq 4$. We then use Lemma 3.3 with $r_1 = a$, $r_2 = 3$, and $b \in C_3$ to obtain: $2\alpha_{a,3} + \alpha_i + \alpha_j = 0 \ \forall i, j \geq 4$. We thus deduce that the coefficients α_i are all equal to $-\alpha_{a,3}$. We then show $\alpha_{1,2} = \alpha_{a,3}$ by using Lemma 3.3 with $1 \in C_1$, $2 \in C_2$ and $3 \in C_3$.

We now have to show $\alpha_{c,i} = \alpha_{d,i} = 0$ for all $c, d \in \{1, 2, 3\}$ and $i \geq 4$. The transformation $\mathcal{T}(\{c, i\}, d, i)$ shows $\alpha_{c,i} = \alpha_{d,i}$. Then the transformation $\mathcal{T}(\{a, 3\}, i, a)$ with $i \geq 4$ yields $\alpha_{a,i} = 0$.

Eventually, we prove that $\alpha_{b,3} = 0$ by considering the transformation $\mathcal{T}(\{1, 2, 3\}, i, 3)$ with $i \geq 4$. \square

4.4. Lower representative inequalities

Theorem 4.5. *If $P_{n,K}$ is full-dimensional the inequalities $x_u + \sum_{i=1}^{u-1} x_{i,u} \geq 1$ for all $u \geq 4$ are facet-defining.*

Proof. We assume that the face $F_u = \{x \in P_{n,K} | x_u + \sum_{i=1}^{u-1} x_{i,u} = 1\}$ is included in a hyperplane $H = \{x \in \mathbb{R}^{|E|+|V|-3} | \alpha^T x = \alpha_0\}$ and we show $\alpha^T x = \alpha_u x_u + \sum_{i=1}^{u-1} \alpha_{i,u} x_{i,u}$.

We again deduce similarly to the proof of Theorem 4.1 that $\alpha_{i,j} = 0 \ \forall i, j \in \{ij \in E \setminus \{12, 13, 23\} | i \neq u, j \neq u\}$, $\alpha_{1,2} = \alpha_{1,3} = \alpha_{2,3} \stackrel{def}{=} \beta$, $\alpha_i = -2\beta \ \forall i \neq u$.

Let c and d be two vertices lower than u . It is then possible through the transformation $\mathcal{T}(\{c, u\}, d, u)$ (Figure 9) to show that $\alpha_{c,u}$ and $\alpha_{d,u}$ are equal to a value that we denote as γ . The two transformations $\mathcal{T}(\{1, 2\}, u, 2)$ and $\mathcal{T}(\{1, 2, 3\}, u, 2)$ (Figures 10 and 11) lead respectively to $\beta + \alpha_u = \gamma$ and $2\beta + \alpha_u = \gamma$. Therefore, $\beta = 0$ and $\alpha_u = \gamma$.

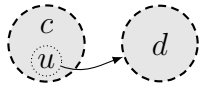


Figure 9:
 $\mathcal{T}(\{c, u\}, d, u)$

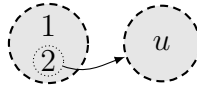


Figure 10:
 $\mathcal{T}(\{1, 2\}, u, 2)$

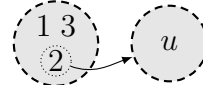


Figure 11:
 $\mathcal{T}(\{1, 2, 3\}, u, 2)$

We eventually have to prove that $\alpha_{i,u} = 0$ for all i greater than u thanks to the transformation $\mathcal{T}(\{c, u\}, \{d, i\}, u)$. \square

Theorem 4.6. *If $P_{n,K}$ is full-dimensional the inequality $x_{1,2} + x_{1,3} + x_{2,3} - \sum_{i=4}^n x_i \geq 3 - K$ for $a \in \{1, 2\}$ - which corresponds to $x_3 + x_{1,3} + x_{2,3} \leq 1$ - is facet-defining.*

Proof. The proof is similar to the one of Theorem 4.4 by considering that a is either 1 or 2. The only difference is that the last transformation ($\mathcal{T}(\{1, 2, 3\} \setminus \{i\}\{3\})$) does not have to be considered. \square

4.5. Triangle inequalities

Theorem 4.7. *If $P_{n,K}$ is full-dimensional the inequality $x_{s,t_1} + x_{s,t_2} - x_{t_1,t_2} \leq 1$ for s, t_1, t_2 distinct in V is facet-defining if and only if the following conditions are satisfied*

- (i) $s < t_1$ or $s < t_2$;
- (ii) $\{s, t_1, t_2\} \neq \{1, 2, 3\}$.

Since the triangle inequalities are a special case of the 2-partition inequalities the reader can refer to Theorem 6.4 for the proof.

5. 2-chorded cycle inequalities

In this section we address the 2-chorded cycle class of inequalities, first introduced in [7]. Let $C = \{e_1, \dots, e_{|C|}\}$ be a cycle in E such that $e_i = c_i c_{i+1}$ for all i in $\{1, 2, \dots, |C| - 1\}$ and $e_{|C|} = c_1 c_{|C|}$. Let $V_C \stackrel{\text{def}}{=} \{c_1, c_2, \dots, c_{|C|}\}$ and $U \stackrel{\text{def}}{=} V \setminus V_C$. Let $u_1, u_2, \dots, u_{|U|}$ be the vertices of U ordered such that $u_1 < u_2 < \dots < u_{|U|}$. From now on all the indices used on a vertex in V_C are given modulo $|C|$ (e.g. $c_{|C|+2}$ corresponds to c_2). The set of 2-chords of C is defined as $\overline{C} = \{c_i c_{i+2} \in E \mid i = 1, \dots, |C|\}$. The 2-chorded cycle inequality induced by a given cycle C of length at least 5 and its corresponding \overline{C} is defined as

$$x(E(C)) - x(E(\overline{C})) \leq \lfloor \frac{1}{2} |C| \rfloor. \quad (21)$$

We skip the proof of the following lemma. The reader can refer to [7] for further details.

Lemma 5.1. *The 2-chorded cycle inequality (21) induced by a cycle C of length at least 5 is valid for $P_{n,K}$. The corresponding face F_C is not facet-defining if $|C|$ is even.*

Theorem 5.2. *The face F_C induced by an odd cycle C of size $2p + 1$ is facet-defining if the following conditions are satisfied:*

- (i) $P_{n,K}$ is full-dimensional;
- (ii) $|U \cap \{1, 2, 3\}| \geq 2$;
- (iii) $2 \leq p \leq n - K - |U \cap \{1, 2, 3\}|$;
- (iv) $K \geq 4$.

Proof. We first highlight K -partitions $\pi_i = \{C_1, C_2, \dots, C_K\}$ in F_C whenever conditions (i) to (iv) are satisfied.

Let c_i be a vertex in V_C . The K -partition π_i is constructed as follows:

- The first cluster contains c_i , u_1 and u_2 ;
- The $2p$ remaining vertices of V_C are scattered in the next clusters such that p edges of C are activated and none of \overline{C} . If cluster K is reached the vertices are distributed in C_{K-1} and C_K (see example Figure 12):
 - $C_q = \{c_{i+2q-1}, c_{i+2q}\} \quad \forall q \in \{2, \dots, \min(p, K)\}$;
 - $C_{K-1} \supset \{c_{i+K+2q-1}, c_{i+K+2q}\}$, for all q odd natural number such that $K + 2q \leq 2p$;
 - $C_K \supset \{c_{i+K+2q-1}, c_{i+K+2q}\}$, for all q even natural number such that $K + 2q \leq 2p$.
- The remaining elements of U are scattered in the next clusters or in C_K :
 - $C_{p+q-1} = \{u_q\} \quad \forall q \in \{3, K - p + 1\}$;
 - $C_K \supset \{u_q\} \quad \forall q \in \{K - p + 2, |U|\}$.

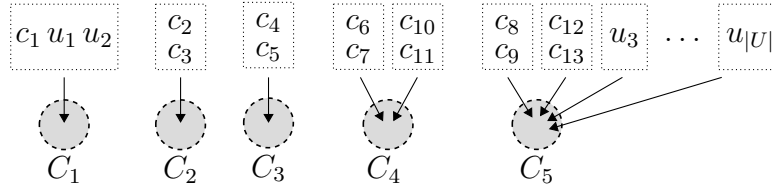


Figure 12: Construction of π_1 if $K = 5$ and $|C| = 13$.

From this construction we directly deduce that π_i is a K -partition. Since no edge of \overline{C} is activated and p edges of C are, π_i is also in F_C .

All the K -partitions considered throughout this proof can be obtained from π thanks to valid transformations (*i.e.* transformations which lead to a K -partition which is still in F_C).

Assume that F_C is included in the hyperplane induced by $\alpha^T x = \alpha_0$. We know from condition (ii) that u_1 and u_2 are in $\{1, 2, 3\}$.

We first show that for all i in $\{1, 2, \dots, |C|\}$: $\alpha_{c_i, c_{i+1}} = -\alpha_{c_i, c_{i+2}} \stackrel{\text{def}}{=} \beta$. To that end we consider the transformations represented Figures 13 and 14. These transformations are valid since they do not alter the number of clusters and the sum $x(E(C)) - x(E(\bar{C}))$. Through these transformations no representative variable is modified since u_1 and u_2 are lower than or equal to 3. We obtain $\alpha_{u_2, c_i} = \alpha_{c_i, c_{i+1}} + \alpha_{c_i, c_{i+2}} + \alpha_{u_1, c_i}$ and $\alpha_{u_1, c_i} = \alpha_{c_i, c_{i+1}} + \alpha_{c_i, c_{i+2}} + \alpha_{u_2, c_i}$. We deduce from them:

$$\lambda_i \stackrel{\text{def}}{=} \alpha_{u_1, c_i} = \alpha_{u_2, c_i}, \quad (22)$$

and

$$\beta_i \stackrel{\text{def}}{=} \alpha_{c_i, c_{i+1}} = -\alpha_{c_i, c_{i+2}}. \quad (23)$$

If we substitute c_i and c_{i+2} in the transformations we get that $\beta_{i+1} = -\alpha_{c_i, c_{i+2}} = \beta_i$. By applying similarly these two transformations on every possible values of i we conclude that all the β_i are equal to a value that we denote by β .

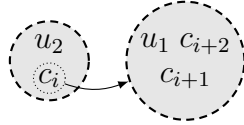


Figure 13:

$$\mathcal{T}(\{c_i, u_2\}, \{c_{i+1}, c_{i+2}, u_1\}, c_i)$$

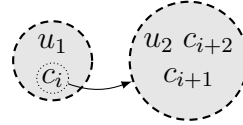


Figure 14:

$$\mathcal{T}(\{c_i, u_1\}, \{c_{i+1}, c_{i+2}, u_2\}, c_i)$$

We now show that all the other coefficients of H are equal to zero.

The transformations $\mathcal{T}(\{u_1, c_i\}, \{c_{i+1}, c_{i+2}\}, c_{i+1})$ and $\mathcal{T}(\{u_1, u_2, c_i\}, \{c_{i+1}, c_{i+2}\}, \{c_{i+1}\})$ respectively show $\alpha_{c_{i+1}} \mathbb{1}(c_{i+1} < c_{i+2}) = \lambda_{i+1} + \alpha_{c_{i+2}} \mathbb{1}(c_{i+1} < c_{i+2})$ and $\alpha_{c_{i+1}} \mathbb{1}(c_{i+1} < c_{i+2}) = 2\lambda_{i+1} + \alpha_{c_{i+2}} \mathbb{1}(c_{i+1} < c_{i+2})$. Accordingly, we have

$$\lambda_i = 0. \quad (24)$$

Therefore, $\alpha_{c_{i+1}} \mathbb{1}(c_{i+1} < c_{i+2}) = \alpha_{c_{i+2}} \mathbb{1}(c_{i+1} < c_{i+2})$. This enables to deduce that all the representative variables from V_C are equal to a constant that we call γ . The transformation $\mathcal{T}(c_i, \{u_1, u_2, c_{i+1}, c_{i+2}\}, u_1)$ then give $\alpha_{u_1, u_2} + \gamma = 0$.

- If $|C \cap \{1, 2, 3\}| = 1$ we directly have $\gamma = 0$ and thus $\alpha_{u_1, u_2} = 0$.

- If $|C \cap \{1, 2, 3\}| = 0$ there exists u_3 in $\{1, 2, 3\} \cap (U \setminus \{u_1, u_2\})$ and the transformation $\mathcal{T}(\{u_1, u_2, u_3, c_i\}, \{c_{i+1}, c_{i+2}\}, u_1)$ yields the same result.

Let t be an index in $\{3, 4, \dots, 2p-3\}$. We now prove that for all i in $\{1, 2, \dots, |C|\}$: $\alpha_{c_i, c_{i+t}} = \alpha_{c_i, c_{i+t+1}} = 0$. The transformations represented in Figure 15 and 16 give

$$\alpha_{c_i, c_{i+t}} + \alpha_{c_i, c_{i+t+1}} = 0, \quad (25)$$

and

$$\alpha_{c_i, c_{i+t}} + \alpha_{c_{i-1}, c_{i+t}} = 0. \quad (26)$$

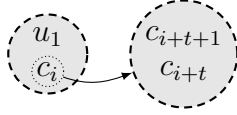


Figure 15:

$$\mathcal{T}(\{c_i, u_1\}, \{c_{i+t}, c_{i+t+1}\}, c_i)$$

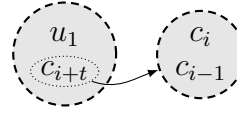


Figure 16:

$$\mathcal{T}(\{c_{i+t}, u_1\}, \{c_{i-1}, c_i\}, c_{i+t})$$

From these two equations we conclude that there exists a variable α_{odd} such that for each $i \in \{1, 2, \dots, |C|\}$ and for all $t \in \{3, 4, \dots, 2p-2\}$

$$\alpha_{c_i, c_{i+t}} = \begin{cases} \alpha_{odd} & \text{if } t+i \text{ is odd} \\ -\alpha_{odd} & \text{otherwise} \end{cases}. \quad (27)$$

Accordingly, we have $\alpha_{c_{2p-2}, c_{2p+1}} = \alpha_{odd}$. However $c_{2p+1} = c_1$ since $|C| = 2p+1$. As a result $\alpha_{c_{2p-2}, c_{2p+1}} = \alpha_{c_1, c_{2p-2}}$ and from the fact that $2p-2$ is even we also get that $\alpha_{c_{2p-2}, c_{2p+1}} = -\alpha_{odd}$. The coefficient α_{odd} is then equal to zero.

To finish the proof we show that if there exists u in $U \cap (V \setminus \{1, 2, 3\})$ then all the coefficients related to u are equal to zero. The transformations presented in Figures 17, 18 and 19 yields, respectively, $\alpha_{c_i, u} = 0$, $\alpha_{u_1, u} = \alpha_u$ and $\alpha_u \mathbb{1}(c_i < u) = 0$. If there exists $c_i < u$ we then obtain that α_u and $\alpha_{u_1, u}$ are equal to zero. If not the transformation $\mathcal{T}(\{c_i, u_1, u_2, u\}, \{c_{i+1}, c_{i+2}\}, u_1)$ gives the expected result.

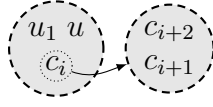


Figure 17:

$$\mathcal{T}(\{u_1, u, c_i\}, \{c_{i+1}, c_{i+2}\}, c_i)$$

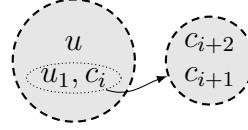


Figure 18:

$$\mathcal{T}(\{u_1, u, c_i\}, \{c_{i+1}, c_{i+2}\}, \{u_1, c_i\})$$

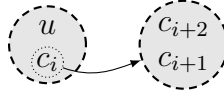


Figure 19:

$$\mathcal{T}(\{c_i, u\}, \{c_{i+1}, c_{i+2}\}, c_i)$$

□

6. 2-Partition inequalities

This section is dedicated to the study of the 2-partition inequalities, first introduced in [7] for the general clique partitioning problem. For two disjoint nonempty subsets S and T of V are defined as

$$x(E(S), E(T)) - x(E(S)) - x(E(T)) \leq \min(|S|, |T|). \quad (28)$$

Let $F_{S,T}$ be the face of $P_{n,K}$ defined by equation 28.

The proof of the three following lemmas is skipped. For further details the reader may refer to [7] for Lemmas 6.1 and 6.2 and to [18] for Lemma 6.3.

Lemma 6.1. *Inequality (28) is valid for $P_{n,K}$.*

Lemma 6.2. *If $|S| = |T|$ $F_{S,T}$ is not a facet of $P_{n,K}$.*

Lemma 6.3. *Given two disjoint subsets S and T of V such that $|S| < |T|$. A K -partition $\pi = \{C_1, C_2, \dots, C_K\}$ is included in $F_{S,T}$ if and only if for all $i \in \{1, 2, \dots, K\}$ $|T \cap C_i| - |S \cap C_i| \in \{0, 1\}$*

Let $U = \{u_1, u_2, \dots, u_{|U|}\}$ be the set of vertices defined by $V \setminus (S \cup T)$ such that $u_1 < u_2 < \dots < u_{|U|}$. The elements of S and T - $\{s_1, s_2, \dots, s_{|S|}\}$ and $\{t_1, t_2, \dots, t_{|T|}\}$ - are similarly sorted.

Theorem 6.4. *If $P_{n,K}$ is full-dimensional, the 2-partition inequality (28) is facet-defining for two non empty disjoint subsets S and T of V if and only if the following conditions are satisfied:*

- (i) $|T| - |S| \in \{1, 2, \dots, K - 1\}$;
- (ii) $|S| \leq n - (K + 2)$;
- (iii) $\forall s \in S \exists t \in T, t > s$;
- (iv) if $|S| = 1 \exists u \in U \cap \{1, 2, 3\}$.

To ensure that the each transformation used throughout the proof of this theorem leads from one K -partition in $F_{S,T}$ to another, we present sufficient conditions on the two clusters involved.

Lemma 6.5. *Let C_1 and C_2 be two disjoint subsets of V . There exists a K -partition in $F_{S,T}$ which contains C_1 and C_2 if*

- (i) $|C_1 \cap T| - |C_1 \cap S| = 0$;
- (ii) $|C_2 \cap T| - |C_2 \cap S| = 1$;
- (iii) $|(C_1 \cup C_2) \cap (T \cup U)| \leq 4$
- (iv) S and T satisfy the conditions of Theorem 6.4.

Proof. Let $S_{1/2}$, $T_{1/2}$ and $U_{1/2}$ refer to the vertices included in C_1 and C_2 which are respectively in S , T and U (i.e.: $S_{1/2} = S \cap (C_1 \cup C_2)$). Let S' (resp. T' and U') correspond to the elements of S (resp. T and U) which are not in C_1 or C_2 (i.e.: $S' = S \setminus S_{1/2}$). We define the elements of S' , T' and U' as follows: $S' = \{s'_1, \dots, s'_{|S'|}\}$, $T' = \{t'_1, \dots, t'_{|T'|}\}$ and $U' = \{u'_1, \dots, u'_{|U'|}\}$.

We first define the partition π as follows (see example in Figure 20):

- The first two clusters are C_1 and C_2 ;
- The next $|T'| - |S'|$ clusters are reduced to one vertex of T' : $C_i = \{t'_{i-2}\} \forall i \in \{3, 4, \dots, |T'| - |S'| + 2\}$;
- The remaining vertices of T' and S' are scattered in the next clusters, or in C_K if it is reached:
 - $C_i = \{s'_{i-|T'|+|S'|+2}, t'_{i-2}\} \forall i \in \{|T'| - |S'| + 3, \dots, \min(|T'| + 2, K - 1)\}$;
 - $C_K \supset \{s'_{i-|T'|+|S'|+2}, t'_{i-2}\} \forall i \in \{K, |T'| + 2\}$;
- The elements of U' are similarly scattered in the next clusters or in C_K :
 - $C_i = \{u'_{i-|T'|+2}\} \forall i \in \{|T'| + 3, \dots, \min(|T'| + |U'| + 2, K - 1)\}$;

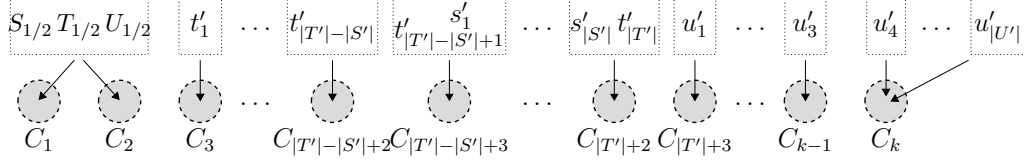


Figure 20: Construction of π if $K = |T'| + 6$ and $|U'| \geq 4$.

$$- C_K \supset \{u'_{i-|T'|-2}\} \quad \forall i \in \{K, |T'| + |U'| + 2\};$$

If the obtained partition π is a K -partition, Lemma 6.3 ensures that it is in $F_{S,T}$. We must therefore prove that $|T'| + |U'| + 2$ is greater than or equal to K .

According to the second condition of Theorem 6.4, K is lower than or equal to $n - (|S| + 2)$. Due to the fact that $n - |S|$ is equal to $|T| + |U|$, we obtain that K must be lower than or equal to $|T_{1/2}| + |T'| + |U_{1/2}| + |U'| - 2$. From the third condition of the current lemma, we deduce that $|T'| + |U'| + 2$ is greater than or equal to K .

We then conclude from Lemma 6.3 that π is in $F_{S,T}$. \square

Each transformation considered in the proof of the theorem involves a couple of clusters which satisfies the conditions of the previous lemma before and after the transformation. Hence, the relations highlighted by the transformations are satisfied by the coefficients of any hyperplane which contains $F_{S,T}$.

Lemma 6.6. *If S and T fulfill the conditions of Theorem 6.4, each hyperplane $H = \{x \in \mathbb{R}^{|E|+|V|-3} \mid \alpha^T x = \alpha_0\}$ which includes $F_{S,T}$ satisfies: $\forall u \in U \quad \forall v \in V \setminus \{u\} \quad \alpha_{u,v} = 0$.*

Proof. To prove this lemma, we consider the following cases:

- case 1: $\{t_1, t_2\} \subset \{1, 2, 3\}$;
- case 2: $\{s_1, s_2\} \subset \{1, 2, 3\}$;
- case 3: $\{u_1, u_2\} \subset \{1, 2, 3\}$;
- case 4: $\{s_1, t_1, u_1\} = \{1, 2, 3\}$.

In each of these cases, we prove that for all $s \in S$, $t \in T$ and $u' \in U$ the coefficients $\alpha_{s,u}$, $\alpha_{t,u}$ and $\alpha_{u,u'}$ are equal to zero.

Lemma	C_1	C_2	Result
3.1	$\{s, t_1\}$	$\{t_2, u\}$	$\alpha_{s,u} = 0 \quad \forall s \in S$ (29)
3.1	$\{s, t_1, u\}$	$\{t_2, u'\}$	$\alpha_{u,u'} = 0 \quad \forall u' \in U$ (30)

Table 2: Results obtained in the case $\{t_1, t_2\} \subset \{1, 2, 3\}$.

Lemma	C_1	C_2	Result
3.1	$\{s_1, t, t'\}$	$\{s_2, t'', u\}$	$\alpha_{t,u} = 0 \quad \forall t \in T, \quad t', t'' \in T \setminus \{t\}$ (31)
3.4	$\{u, t\}$	$\{u'\}$	$\alpha_{u,u'} = \alpha_{t,u'} \quad \forall u' \in U, \quad t < u', u < u'$ (32)
3.4	$\{s_1, t, u\}$	$\{t', u'\}$	$\alpha_{s_1,u'} = \alpha_{u,u'} \quad \forall u' \in U, \quad u < u' < t'$ (33)

Table 3: Results obtained in the case $\{s_1, s_2\} \subset \{1, 2, 3\}$.

Case 1: $\{t_1, t_2\} \subset \{1, 2, 3\}$

If the two first vertices of T are lower than 4 the results shown table 2 yield the major part of the lemma. It remains to prove that $\alpha_{t,u} = 0$ for all $t \in T$.

From condition (i) and (ii) we can deduce that $|T| \leq n - 3$. Thus, we either have $|S| \geq 2$ or $|U| \geq 2$. Let C be a cluster equal to $C = \{s_2, t'\}$ with $t' \in T \setminus \{t_1, t\}$ if $|S| \geq 2$ and $C = \{u'\}$ with $u' \in U \setminus \{u\}$ if $|U| \geq 2$. The transformations $\mathcal{T}(\{t_2, u\}, C, \{u\})$ and $\mathcal{T}(\{s_1, t_1, t_2, u\}, C, \{u\})$ lead to two identical equality except for the coefficients $\alpha_{t,u}$ and $\alpha_{s_1,u}$ which appear in the second. Since $\alpha_{s_1,u}$ is equal to zero the same applies to $\alpha_{t,u}$.

Eventually, we obtain $\alpha_{t_1,u} = 0$ via transformation $\mathcal{T}(\{s, t_1\}, \{t_2, u\}, \{u\})$.

Case 2: $\{s_1, s_2\} \subset \{1, 2, 3\}$

According to (i), since $|S| \geq 2$, $|T| \geq 3$. Then the first line of table 3 proves that $\alpha_{t,u}$ is always equal to zero for all $t \in T$. Let s be a vertex in $S \setminus \{s_1\}$. The transformations $\mathcal{T}(\{s_1, t, u\}, \{t'\}, \{u\})$ and $\mathcal{T}(\{s_1, s, t, t'', u\}, \{t'\}, \{u\})$ show that $\alpha_{s,u}$ is equal to zero. A symmetrical reasoning by substituting respectively s_1 and s by s_2 and s_1 leads to $\alpha_{s_1,u} = 0$. Eventually, equations (32) and (33) show that all the coefficients $\alpha_{u,u'}$ are also equal to zero.

Lemma	C_1	C_2	Result
3.1	$\{s_1, t, u\}$	$\{t_1, u'\}$	$\alpha_{u,u'} = 0 \quad \forall u' \in U$ (34)
3.1	$\{t, u_1, u\}$	$\{t_1, s\}$	$\alpha_{s,u} = 0 \quad u \neq u_1$ (35)
3.1	$\{u_1, u\}$	$\{s_1, t, t'\}$	$\alpha_{t,u} = 0 \quad \forall t, t' \in T, u \neq u_1$ (36)

Table 4: Results obtained in the case $\{s_1, t_1, u_1\} = \{1, 2, 3\}$.

Case 3: $\{u_1, u_2\} \subset \{1, 2, 3\}$

Let u be a vertex of U and $a \in \{u_1, u_2\} \setminus \{u\}$. The transformations $\mathcal{T}(\{a\}, \{t, u\}, \{u\})$ and $\mathcal{T}(\{a, s, t\}, \{t, u\}, \{u\})$ lead to : $\alpha_{s,u} + \alpha_{t,u} = 0$. Then, $\mathcal{T}(\{s, t\}, \{a, t'\}, \{a\})$ and $\mathcal{T}(\{s, t\}, \{a, t', u\}, \{a, u\})$ show that $\alpha_{t,u}$ is equal to zero. Thus, the same applies to $\alpha_{s,u}$.

We now have to prove that $\alpha_{u,u'}$ is equal to zero for all $u, u' \in U$. We assume wlog that u is lower than u' . If $\min(s_1, t_1) \leq 3$, $\mathcal{T}(\{u, u'\}, \{s_1, t_1, t_2\}, \{u'\})$ gives the result. Otherwise, U necessarily contains at least three vertices. Let $a \in U \setminus \{u, u'\}$. We then conclude through $\mathcal{T}(\{a, u\}, \{t\}, \{a\})$ and $\mathcal{T}(\{a, u, u'\}, \{t\}, \{a, u'\})$.

Case 4: $\{s_1, t_1, u_1\} = \{1, 2, 3\}$

Table 4 enable to conclude for all the coefficients except α_{s,u_1} for all $s \in S$ and α_{t,u_1} for all $t \in T$.

$\mathcal{T}(C \cup \{t_1\}, \{t\}, \{t_1, t\})$ with C successively equal to $\{s_1\}$ and $\{s_1, u_1\}$ shows that α_{t_1, u_1} and α_{t, u_1} are equal. Then, $\mathcal{T}(\{t_1, u_1\}, \{s_1, t\}, \{u_1\})$ leads to: $\alpha_{s_1, u_1} = 0$.

As previously stated, at least one of the set S ou U contains more than one vertex. If U contains at least two vertices, $\mathcal{T}(P \cup \{s_1, t\}, \{u_2\}, \{s_1, t\})$ with P successively equal to $\{t_1\}$ and $\{t_1, u_1\}$ gives: $\alpha_{t, u_1} = 0$. If S contains more than one vertex, we consider $\mathcal{T}(P \cup \{s, t\}, \{s', t'\}, \{s, t\})$ with $\{s, s'\} \subset S$, $\{t, t'\} \subset T$ and P a set of vertices. By first taking s equal to s_1 and considering P equal to $\{t_1\}$ and $\{t_1, u_1\}$, we show that α_{t, u_1} is equal to zero. The same reasoning with $s \in S \setminus \{s_1\}$ gives: $\alpha_{s_1, u_1} = 0$. □

We now present the proof of Theorem 6.4.

Proof. We start by proving that if any of the conditions is not satisfied the 2-partition inequality is not facet-defining.

- (i) Lemmas 6.2 and 6.3 respectively lead to $|T| - |S| > 0$ and $|T| - |S| \leq K$. It remains to prove that if $|T| - |S| = K$, equation (28) is not facet-defining. In that case, each cluster C verifies, $|C \cap T| = |C \cap S| + 1$ and $F_{S,T}$ is thus included in the $|T|$ hyperplanes defined by $\sum_{i \in T \setminus \{t\}} x_{i,t} = \sum_{i \in S} x_{i,t}$ for all $t \in T$.
- (ii) We get via Lemma 6.3 that each cluster C contains at least as many vertices from T than from S . Thus, at least $|S|$ vertices are not a representative of their cluster, and then $K \leq n - |S|$. If $K = n - |S|$, $x(E(S), E(T))$ is equal to $|S|$ since the only edges in the partition are necessarily the ones which ensure that each vertex $s \in S$ is linked to a vertex in T . Eventually, if $K = n - |S| - 1$ we deduce by enumerating all possible configurations that $F_{S,T}$ is included in the hyperplane defined by $x(E(U)) + x(E(U), E(T)) + x(E(S), E(T)) - x(E(S)) = 3$.
- (iii) Lemma 6.3 states that each cluster contains at least as much vertices from T than from S . As a result if there was a vertex s in S greater than any vertex in T we would have $x_s = 0$ since s would never be a representative.
- (iv) If $|S| = 1$ and there is no element of U in $\{1, 2, 3\}$ there is then at least two elements of T , t_1 and t_2 , whose indices are lower than or equal to 3. The last element in $\{1, 2, 3\}$ is either in T or S .
 - If it is in T we have $\sum_{i=4}^n x_i = x_{1,2} + x_{1,3} + x_{2,3} + K - 3$;
 - If it is in S we have $\sum_{i=4}^n x_i = x_{s,t_1} + x_{s,t_2} + K - 3$.

To prove that $F_{S,T}$ is facet-defining under the above mentioned conditions, we assume that $F_{S,T}$ is included in a hyperplane $H = \{x \in \mathbb{R}^{|E|+|V|-3} \mid \alpha^T x = \alpha_0\}$. We highlight relations between the α coefficients thanks to several transformations from one K -partition to another in $F_{S,T}$.

We consider three cases.

- case 1: $|S| = 1$;
- case 2: $|S| \geq 2$ and $|U| = 0$;
- case 3: $|S| \geq 2$ and $|U| \geq 1$.

Case 1: $|S| = 1$

In that case, according to (iv), u_1 is in $\{1, 2, 3\}$. From (i) and (ii) we deduce that $|T|$ is lower than $n - 3$. As a result, since $|S| = 1$, $|U|$ is greater than 2.

Let t be either t_1 or t_2 . For all $t' \in T \setminus \{t\}$, by noticing that $\min\{s_1, t, u_2\} \leq 3$, we deduce, from $\mathcal{T}(\{s_1, t, u_2\}, \{t', u_1\}, \{u_1\})$ that: $\alpha_{t'} = 0$.

Thanks to (iii) we know that s_1 is lower than $t_{|T|}$. Moreover, since only u_1 and s_1 may be lower than $\min(t_1, u_2)$, we know that $\alpha_{\min(t_1, u_2)}$ is equal to zero. Thus, $\mathcal{T}(\{t_1, u_2\}, \{s_1, t_{|T|}, u_1\}, \{u_1\})$ gives: $\alpha_{s_1} = 0$. For all $u \in U \setminus \{u_1\}$, the transformation $\mathcal{T}(\{u_1, u\}, \{t_{|T|}\}, \{u_1\})$ proves that the coefficients α_u are null.

Eventually, for all $t, t' \in T$, we prove that the expressions $\alpha_{s_1, t}$ and $-\alpha_{t, t'}$ are equal through $\mathcal{T}(\{s_1, t\}, \{t'\}, \{s_1\})$ and $\mathcal{T}(\{s_1, t, t'\}, \{u_1\}, \{t\})$.

Case 2: $|S| \geq 2$ and $|U| = 0$

Conditions (i) and (ii) still lead to $|T| \leq n - 3$ which gives $|S| \geq 3$, in the current case. Since $P_{n, K}$ is only full-dimensional for values of K greater than two, condition (ii) implies that $|T|$ is greater than four.

In this part of the proof, let the expressions \bar{t} and \tilde{t} both correspond to t_1 or t_2 with the restriction that \bar{t} and \tilde{t} are distinct. Similarly, \bar{s} and \tilde{s} correspond to s_1 or s_2 . Since $|U| = 0$, $\min(\bar{t}, \bar{s})$ is lower than four and its corresponding representative variable $\alpha_{\min(\bar{t}, \bar{s})}$ is, therefore, equal to zero.

We first prove $\forall s \in S \setminus \{s_1, s_2\} \forall t, t' \in T \setminus \{t_1, t_2\}$ that $\beta \stackrel{def}{=} \alpha_{s, t} = -\alpha_{t, t'} = \alpha_{\bar{s}, t_{|T|}} = -\alpha_{\tilde{t}, t_{|T|}}$.

We obtain $\alpha_{s, t} = -\alpha_{t, t'}$ thanks to the transformations represented Figures 21 and 22. The remaining part of the equation is highlighted via transformations $\mathcal{T}(\{s, t, t_{|T|}\}, \{s', t'\}, t_{|T|})$ and $\mathcal{T}(\{\bar{s}, \bar{t}, s, t, t_{|T|}\}, \{s', t'\}, t_{|T|})$.

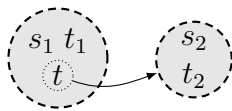


Figure 21:

$$\mathcal{T}(\{s_1, t_1, t\}, \{s_2, t_2\}, t)$$

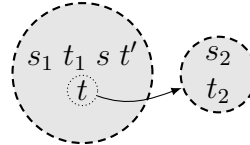


Figure 22:

$$\mathcal{T}(\{s_1, t_1, s, t, t'\}, \{s_2, t_2\}, t)$$

For any couple of vertex $(s, t) \in S \times T$, we now consider the transformations $\mathcal{T}(\{\bar{s}, s, \bar{t}, t\}, \{t_{|T|}\}, \{s\})$ and $\mathcal{T}(\{\bar{s}, s, \bar{t}, \tilde{t}\}, \{t\}, \{\bar{t}, t\})$ which respectively lead to

$$\alpha_{s, \bar{s}} + \alpha_{s, \bar{t}} + \alpha_{s, t} + \alpha_{t_{|T|}} = \alpha_s + \beta \quad (37)$$

and

$$\alpha_{\tilde{t}, \tilde{t}} + \alpha_{\tilde{s}, \tilde{t}} + \alpha_{s, \tilde{t}} + \alpha_t = \alpha_{\tilde{t}} + \alpha_{s, t} + \alpha_{\tilde{s}, t} + \alpha_{\tilde{t}, t}. \quad (38)$$

Equation (37) shows that for any given $s \in S$, the value of $\alpha_{s, t}$ is the

same for all $t \in T \setminus \{t_{|T|}\}$. Since α_{s,t_3} is equal to β , we deduce:

$$\alpha_{s,\bar{t}} = \beta \quad \forall s \in S. \quad (39)$$

After replacing $\alpha_{s,\bar{t}}$ by β in equation (38), a similar reasoning can be applied to prove: $\alpha_{\bar{s},t} = \beta \quad \forall t \in T \setminus \{t_1, t_2\}$.

The transformation $\mathcal{T}(\{\bar{s}, \tilde{s}, s, \bar{t}, \tilde{t}, t\}, \{t_{|T|}\}, \{s\})$ and equation (37) give: $\alpha_{s,\bar{s}} = -\beta \quad \forall s \in S \setminus \{s_1, s_2\}$.

If $t_3 \leq 3$ we prove by symmetry that $\alpha_{\bar{t},t} = \alpha_{t_3,t} \quad \forall t \in T \setminus \{t_1, t_2, t_3\}$. Thus, $\alpha_{\bar{t},t} = \beta$. Otherwise $s_1 \leq 3$ and the same result is obtained via equation (37) and transformations $\mathcal{T}(\{t, s_1\}, \{t_{|T|}\}, \{s_1\})$ and $\mathcal{T}(\{s, t, \bar{t}\}, \{s, t_{|T|}\}, \{t\})$.

Equation (38) now shows that the value of the representative coefficients α_t for any $t \in T \setminus \{t_1, t_2\}$ are the same. Equation (37) applied on any $s \in S \setminus \{s_1, s_2\}$ proves: $\alpha_s = \alpha_t$. Let γ be this value.

If t_3 or s_3 is lower than four, γ is equal to zero (since $\alpha_{\min(s_3, t_3)}$ is null). Otherwise, s_1 and t_1 must be lower than four. Since $\mathcal{T}(\{\bar{s}, \bar{t}\}, \{t_{|T|}\}, \{\bar{s}\})$ gives $\gamma = \alpha_{\max\{\bar{s}, \bar{t}\}}$ we deduce, in that case also, that α_t is equal to zero. Since $\max\{\bar{s}, \bar{t}\}$ and $\min\{\bar{s}, \bar{t}\}$ are both equal to zero, we conclude that $\alpha_{\bar{s}}$ and $\alpha_{\bar{t}}$ are null.

Eventually, to prove that the value of α_{s_1, s_2} and α_{t_1, t_2} is $-\beta$, we use equations (38) and (37) with $t = t_1$ and $s = s_1$.

Case 3: $|S| \geq 2$ and $|U| \geq 1$

For all s in S and all t, t' distinct in $T \setminus \{t_1\}$, we first prove that $\alpha_{s,t}$, $-\alpha_{t,t'}$ and $-\alpha_{t_1, t_{|T|}}$ are all equal to a constant called β .

Let s' be a vertex in $S \setminus \{s\}$. The transformations represented in Figures 23 and 24 lead to $\alpha_{s,t} = -\alpha_{t,t'}$ and

$$\alpha_{u_1} \mathbb{1}(t < u_1) + \alpha_{t_1, t} + \beta = \alpha_t \mathbb{1}(t < u_1). \quad (40)$$

Equation (40) when t is equal to $t_{|T|}$ can be simplified via $\mathcal{T}(\{u_1\} \{s', t, t_{|T|}\} \{t_{|T|}\})$ in $\alpha_{t_1, t_{|T|}} = -\beta$.

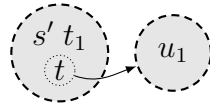


Figure 23: $\mathcal{T}(\{s', t_1, t\}, \{u_1\}, \{t\})$

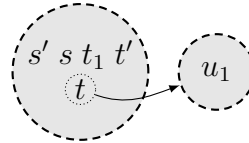


Figure 24:

$\mathcal{T}(\{s', s, t_1, t, t'\}, \{u_1\}, \{t\})$

We now show that for all triplets $(s, t, u) \in S \times T \setminus \{t_1, t_{|T|}\} \times U$ the two higher

vertices of the triplet have the the same representative coefficient equal to $\alpha_{t_{|T|}}$.

The transformation $\mathcal{T}(\{s, t\}, \{t_{|T|}\}, \{s\})$ leads to

$$\alpha_{t_{|T|}} = \alpha_{\max\{s, t\}}. \quad (41)$$

Then the transformation $\mathcal{T}(\{s, t, t_{|T|}\}, \{u\}, \{t\})$ yields $\alpha_t \mathbb{1}\{u < t < s\} + \alpha_u \mathbb{1}\{t < u\} = \alpha_t \mathbb{1}\{s < t < u\} + \alpha_s \mathbb{1}\{t < s\}$ which can be simplified in

$$\alpha_{\max\{u, t\}} = \alpha_{\max\{s, t\}}, \quad (42)$$

using the fact that $\mathbb{1}\{u < t < s\} - \mathbb{1}\{s < t < u\}$ is equal to $\mathbb{1}\{u < t\} - \mathbb{1}\{s < t\}$.

If t is lower than s or u , equations (41) and (42) prove that the two higher vertices among the triplet (s, t, u) have representative variables equal to $\alpha_{t_{|T|}}$. Otherwise, the transformation $\mathcal{T}(\{u\}, \{s, t, t_{|T|}\}, \{s, t\})$ yields $\alpha_{\max\{u, s\}} = \alpha_{t_{|T|}}$ which leads with equation (41) to the same result.

The next of this proof consists in showing that the representative coefficient of any vertex which is not t_1 is equal to zero. This result is true if at least two vertices of (s_1, t_2, u_1) are in $\{1, 2, 3\}$ (since the representative variables of the two greatest representatives are equal to $\alpha_{t_{|T|}}$). It remains to consider the cases in which only one of these three elements are in $\{1, 2, 3\}$. Let x be this element.

- If $x = t_2$, t_3 is necessarily in $\{1, 2, 3\}$. The coefficient $\alpha_{\min(s_1, u_1)}$ is equal to $\alpha_{t_{|T|}}$ and $\mathcal{T}(\{s_2, t_1, t_2\}, \{s_1, t_{|T|}, u_1\}, \{t_2\})$ gives the result.
- If $x = u_1$, u_2 is necessarily in $\{1, 2, 3\}$ and $\mathcal{T}(\{u_1, u_2\}, \{t_2\}, \{u_2\})$ enable to conclude.
- If $x = s_1$, s_2 is necessarily in $\{1, 2, 3\}$. If the third vertex of this set is t_1 , $\mathcal{T}(\{s_2, t_1\}, \{t_{|T|}\}, \{s_2\})$ gives the result. Otherwise, S contains at least three vertices and $(s_1, s_2, s_3) = (1, 2, 3)$. We conclude using $\mathcal{T}(C \cup \{s_1, s_3, t_1, t_3\} \setminus \{t_{|T|}\} \setminus \{s_3\})$ with C successively equal to \emptyset and $\{s_2, t_2\}$ and the transformation $\mathcal{T}(\{s_1, s_2, s_3, t_1, t_3, t_{|T|}\}, \{t_2\}, \{s_1, s_2, t_{|T|}\})$.

Let s be either s_1 or s_2 . If $t_1 \in \{4, \dots, n\}$, we prove that α_{t_1} is null thanks to $\mathcal{T}(\{s, t_1, u_1\}, \{t_{|T|}\}, \{t_1, t_{|T|}\})$ and $\mathcal{T}(\{s_1, s_2, t_1, t_2, u_1\}, \{t_{|T|}\}, \{t_1, t_{|T|}\})$. Eventually, we prove for all $s, s' \in S$ distincts and $t \in T \setminus \{t_1\}$ that expressions α_{s, t_1} , $-\alpha_{s, s'}$ and $-\alpha_{t_1, t}$ are equal to β through: $\mathcal{T}(\{s, t_1\}, \{t_{|T|}\}, \{s\})$, $\mathcal{T}(\{s, s', t_1, t_2\}, \{t_{|T|}\}, \{s\})$ and $\mathcal{T}(\{s_1, t_1, t\}, \{s_2, t_{|T|}\}, \{t\})$. \square

7. General clique inequalities

The *clique inequalities* have been introduced by Chopra and Rao [1] and correspond to the fact that for any m -partition π ($m \leq K$) and any set $Z \subset V$ of size $K + 1$, at least two vertices of Z are necessarily in the same cluster. The clique inequality induced by a given set Z is:

$$x(E(Z)) \geq 1. \quad (43)$$

The general clique inequalities are obtained by increasing the size of both Z and the right-hand side of equation 43. Thus, the general clique inequalities induced by a set $Z \subset V$ of size $qK + r$ with $q \in \mathbb{N}$ and $r \in \{0, 1, \dots, K - 1\}$ is defined by Chopra and Rao as:

$$x(E(Z)) \geq \binom{q+1}{2} r + \binom{q}{2} (K - r). \quad (44)$$

Let P_Z be the face of $P_{n,K}$ defined by equation (44). As represented Figure 25, the lower bound of inequality (44) corresponds to the minimal value of $x(E(Z))$ in the incidence vector of a K -partition – thus ensuring the validity of this inequality. It is obtained by setting $q + 1$ vertices of Z in each of the r first clusters and q vertices in each of the $K - r$ remaining clusters.

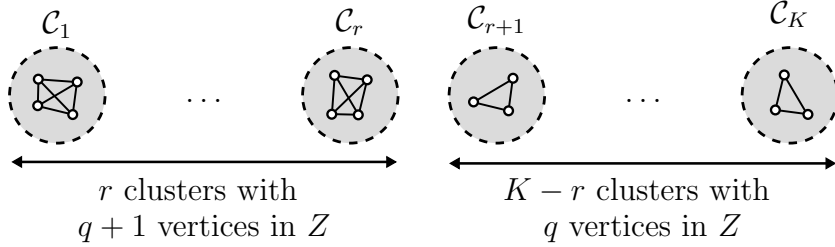


Figure 25: Distribution of Z vertices in a K -partition included in F_Z (case where q is equal to three).

These inequalities have also been studied by Labbé and Öszoy [16] in the case where the clusters must contain at least F_L vertices. In this context, the size of Z must be greater than or equal to $\lfloor \frac{n}{F_L} \rfloor$. Finally, Ji and Mitchell also studied these inequalities that they called *pigeon inequalities* [19].

In the following $U = \{u_1, u_2, \dots, u_{|U|}\}$ is used to denote $V \setminus Z$ such that $u_1 < u_2 < \dots < u_{|U|}$. The vertices in $Z = \{z_1, \dots, z_{K+1}\}$ are similarly sorted.

Theorem 7.1. *If $P_{n,K}$ is full-dimensional, for a given $Z \subset V$ of size $K+1$, inequality (44) is facet-defining if and only if:*

- (i) $|U| \geq 1$ and $u_1 \leq 3$;
- (ii) $z_{|Z|} = n$
- (iii) $|Z| \in \{K+1, \dots, 2K-1\}$.

Lemma 7.2. *Let V_1 and V_2 be two disjoint subsets of V and let Z be a subset of V which satisfies the conditions of Theorem 7.1. Then, there exists a K -partition in F_Z which includes V_1 and V_2 if $\{|V_1 \cap Z|, |V_2 \cap Z|\}$ is equal to $\{1, 2\}$.*

Proof. Given the bounds on the size of Z , q is necessarily equal to one and $r \in \{1, \dots, K-1\}$. Consequently, each K -partition included in F_Z contains at least one cluster with exactly one vertex in Z and at least one cluster with exactly two vertices in Z .

Let $\pi = \{C_1, \dots, C_K\}$ be the K -partition such that:

- $C_1 = V_1$ and $C_2 = V_2$.
- Clusters C_3 to C_{r+1} each contains $q+1$ vertices from Z .
- Clusters C_{r+2} to C_K each contains q vertices from Z .
- The vertices in U which are not included in V_1 or V_2 are in C_K .

This construction is always possible since $|Z|$ is equal to $qK + r$. It can easily be checked – by computing $x^\pi(Z)$ – that π is in F_Z . \square

Each transformation $\mathcal{T}(C_1, C_2, R) \mapsto \{C'_1, C'_2\}$, considered in the proof of Theorem 7.1 is such that the couples (C_1, C_2) and (C'_1, C'_2) satisfy the conditions imposed on V_1 and V_2 in Lemma 7.2. This ensure the validity of the transformations. We now present the proof of Theorem 7.1.

Proof. If the first condition of the theorem is not satisfied, the three first vertices are in Z and cannot be in the same cluster. Consequently P_Z is included in the hyperplane defined by $\sum_{i=4}^n x_i - x_{1,2} - x_{1,3} - x_{2,3} = K-3$. If (ii) is false, the vertices u which are greater than z_{K+1} cannot be representative since each cluster contains at least one element of Z . Thus, P_Z is included in the hyperplanes: $x_u = 0 \ \forall u > z_{|Z|}$. Eventually, if Z contains more than $2K-1$ vertices, each cluster necessarily include at least two vertices from Z .

Thus, $z_{|Z|}$ cannot be a representative and F_Z is included in the hyperplane induced by $x_{z_{|Z|}} = 0$.

Let $H = \{x \in \mathbb{R}^{|E|+|V|-3} \mid \alpha^T x = \alpha_0\}$ be a hyperplane which includes P_Z and let $z_i < z_j < z_k$ be three elements of Z . The transformation $\mathcal{T}(\{z_i, z_k\}, \{z_j\}, \{z_k\})$, first shows: $\alpha_{z_i, z_k} = \alpha_{z_j, z_k}$. Thus, for a given k and for all $j \in \{1, \dots, k-1\}$ the coefficients α_{z_j, z_k} are equal to a constant, referred to as β_k .

For all j and k greater than i , $\mathcal{T}(\{z_i, z_k\}, \{z_j\}, \{z_i\})$ gives

$$\beta_k - z_k = \beta_j - z_j. \quad (45)$$

Let z, z' and z'' be three distincts elements of Z . The transformation $\mathcal{T}(\{u_1, z\}, \{z'\}, \{u_1\})$ leads to $\alpha_{z'} + \alpha_{u_1, z} = \alpha_z + \alpha_{u_1, z'}$. This result and $\mathcal{T}(\{u_1, z\}, \{z', z''\}, \{u_1\})$ give for all $h \in \{2, \dots, K+1\}$: $\alpha_{u_1, z_h} = 0$ and

$$\alpha_{u_1, z_1} + \alpha_{z_h} = \alpha_{z_1}. \quad (46)$$

From equations (45) and (46), we obtain that for all $h \in \{2, \dots, K+1\}$, the representative coefficients of z_h are equal and that the same applies to the β_h .

If $|U|$ is equal to one, the proof is over. Indeed, in that case z_2 is lower than four and thus α_{z_2} is equal to zero, which gives via equation (46) $\alpha_{u_1, z_1} = 0$.

If $|U|$ is greater than two, we then prove that $\alpha_{u, z}$ is equal to zero for all $u \in U \setminus \{u_1\}$ and all $z \in Z$. This is obtained thanks to $\mathcal{T}(\{u_1, u, z_1\}, \{z\}, \{u_1, u\})$, $\mathcal{T}(\{u_1, u, z_1\}, \{z, z'\}, \{u_1, u\})$ and equation (46).

We show that α_{z_1} is equal to α_{z_2} , thanks to $\mathcal{T}(\{u_2, z_2\}, \{z_1\}, \{u_2\})$ which leads through equation (46) to $\alpha_{u_1, z_1} = 0$.

If $|U|$ is equal to two, α_z is equal to zero, and it remains to prove that α_{u_1, u_2} is equal to zero, which can be done by $\mathcal{T}(\{u_1, u_2, z_2\}, \{z_1\}, \{u_2\})$.

Otherwise, for a given u in U , let U' be a subset of $U \setminus \{u\}$ which contains u_1 or u_2 . The transformation $\mathcal{T}(\{u, z, U'\}, \{n\}, \{u\})$ gives

$$\sum_{u' \in U'} \alpha_{u, u'} + \alpha_z = \alpha_u \quad \forall z \in Z. \quad (47)$$

This equation shows that the sum of $\alpha_{u, u'}$ is equal to a constant for any possible U' . Let u' and u'' be two vertices in $U \setminus \{u\}$. By successively choosing U' equal to $\{u''\}$ and $\{u', u''\}$ we obtain: $\alpha_{u, u'} = 0$.

Eventually, equation (47) gives: $\alpha_z = \alpha_u \quad \forall (u, z) \in U \times Z$. Since α_{u_1} is equal to zero, the same applies to the other representative variables. \square

8. Strengthened triangle inequalities

Theorem 4.7 states that inequalities (1) are not facet-defining if s is greater than both t_1 and t_2 . However, they can be strengthened by adding the term x_s to the left side of the inequality whenever s is greater than three (otherwise x_s is equal to zero):

$$x_{s,t_1} + x_{s,t_2} - x_{t_1,t_2} + x_s \leq 1. \quad (2')$$

For three distinct vertices s , t_1 and t_2 , let P_{s,t_1,t_2} be the face of $P_{n,K}$ defined by equation (2').

Theorem 8.1. *Let s, t_1 and t_2 be three vertices in V such that $s > t_2 > t_1$ and $s > 3$. When $P_{n,K}$ is full-dimensional the inequality $x_{s,t_1} + x_{s,t_2} - x_{t_1,t_2} + x_s \leq 1$ is facet-defining if and only if $(t_2 > 3)$ or $(K \leq n - 3)$.*

Proof. Assume that $t_2 \leq 3$ and $K = n - 2$. Let π be a K -partition such that the three first vertices are in the same cluster. As K is equal to $n - 2$ the $K - 1$ other clusters are necessarily reduced to one vertex. Hence the left part of equation (2') is equal to zero and π is not in P_{s,t_1,t_2} . Since 1, 2 and 3 cannot be together we deduce that P_{s,t_1,t_2} is included in the hyperplane defined by $\sum_{i=4}^n x_i - x_{1,2} - x_{1,3} - x_{2,3} = K - 3$.

In the following of the proof the terms t and t' refer to either t_1 or t_2 with the restriction that t is different from t' .

Let $\alpha^T x = \alpha_0$ induce an hyperplan which includes P_{s,t_1,t_2} for $t_2 > 3$ or $K \leq n - 3$ and let $U = \{u_1, u_2, \dots, u_{|U|}\}$ be $V \setminus \{s, t_1, t_2\}$ such that $u_1 < u_2 < \dots < u_{|U|}$. Similarly to the proof of Lemma 6.6 we show that $\alpha_{u,v}$ is equal to zero for all $u \in U$ and all $v \in V \setminus \{u\}$.

Then we consider the transformation $\mathcal{T}(\{C\}, \{t, u_1\}, \{u_1\})$ with $C = \{t', u_2\}$ if $K = n - 2$ and $C = \{t', u_2, s\}$ otherwise. By noting that $\min\{t', u_2\}$ is lower than four, we deduce: $\alpha_t = 0$.

We then prove that α_s is equal to $\alpha_{s,t}$ thanks to $\mathcal{T}(\{t, u_1\}, \{s\}, \{t\})$. The transformation $\mathcal{T}(\{t_1, u_1, u\}, \{s\}, \{t_1, u_1\})$ gives: $\alpha_u = 0, \forall u \in U \setminus \{u_1\}$. Eventually we obtain, through $\mathcal{T}(\{s, t, t'\}, \{u\}, \{t\})$: $\alpha_{t,t'} = -\alpha_{s,t}$. \square

9. Paw inequalities

Given a subset $W = \{a, b, c, d\}$ of V , we define the *paw inequality* associated to W by:

$$x_{a,b} + x_{b,c} - x_{a,c} + x_{c,d} + x_b + x_c \leq 2. \quad (48)$$

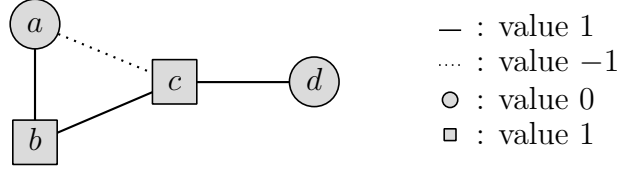


Figure 26: Representation of the coefficients of the paw inequality associated to a subset $\{a, b, c, d\}$ of V .

Figure 26 represents the variables in this inequality.

Lemma 9.1. *Let $K \in \{3, \dots, n-2\}$. Inequality (48) is valid for $P_{n,K}$ if and only if*

1. $a < b$;
2. $\min(b, c, d) = d$.

Proof. If $\min(b, c, d)$ is not d , the left-hand side of equation (48) is equal to 3 for any K -partition with a cluster equal to $\{b, c, d\}$. If b is lower than a , then equation (48) is not satisfied for any K -partition $\pi = \{C_1, \dots, C_K\}$ such that $\{a, b\} \subset C_1$ and $C_2 = \{c\}$.

The addition of the triangle inequality (1)

$$x_{a,b} + x_{b,c} - x_{ac} \leq 1 \quad (49)$$

and the lower representative inequality (2)

$$x_c + x_{c,d} \leq 1 \quad (50)$$

ensures that the paw inequality is valid if x_b is equal to zero.

If x_b is equal to one, we show that (48) is still valid since equation (49) and equation (50) cannot both be tight. In that case, a and d cannot be in the same cluster than b since their indices are lower. The only way for (49) to be tight under these conditions is for b and c to be together. Equation (50) is tight if c is representative or if c and d are together. In both cases x_b cannot be equal to one if b and c are together. \square

Let F_P be the face of $P_{n,K}$ associated to inequality (48).

Lemma 9.2. *Under the conditions of Lemma 9.1 the face F_P is not a facet if $c < b$ or $K = n - 2$.*

Proof. If c is lower than b we prove that F_P is included in the hyperplane induced by $x_c + x_{c,d} = 1$. The expression

$$x_c + x_{c,d} \tag{51}$$

can be equal to 0, 1 or 2.

If expression (51) is equal to 0, the solutions in F_P satisfy : $x_{a,b} + x_{b,c} - x_{a,c} + x_b = 2$. This equation cannot be true since b has to be greater than both a and c according to Lemma 9.1 b and the condition of the current lemma. The expression (51) cannot be equal to two either since d is lower than c .

As a result expression (51) is necessarily equal to one.

If K is equal to $n - 2$, no K -partition can contain both a and c and thus, F_P is included in the hyperplane induced by $x_{a,c}$. □

Theorem 9.3. *Let $K \in \{3, n - 3\}$ and $b \in \{4, \dots, n\}$, F_P is facet defining of $P_{n,K}$ if and only if*

1. $d < b < c$;
2. $a < b$.

Proof. A K -partition containing a cluster equal to $\{b, c\}$ satisfies the paw inequality. Thus, by setting C_3 equal to $\{b, c\}$, one can use the same reasoning as in the proof of Theorem (3.5) to obtain the following relations on the coefficients of an equation $\alpha^T x = \alpha_0$ satisfied by all the points in F_P :

- $\alpha_{i,j} = 0 \ \forall i \in V \setminus \{b, c, 1, 2, 3\} \forall j \in V \setminus \{b, c, i\}$;
- $\alpha_{1,2} = \alpha_{1,3} = \alpha_{2,3} \stackrel{def}{=} \beta$;
- $\alpha_i = -2\beta \ \forall i \in V \setminus \{b, c, 1, 2, 3\}$.

The value of the remaining α coefficients can be obtained through the transformations represented table 5. □

Theorem 9.4. *Let $K \in \{3, n - 3\}$. The face F_P associated to the inequality $x_{a,b} + x_{b,c} - x_{a,c} + x_{c,d} + x_c + x_{1,2} - \sum_{i=4}^n \leq 4 - K$ - which corresponds to the paw inequality (48) for b equal to three - is facet for $P_{n,K}$ if and only if*

1. $d < 3 < c$;
2. $a < 3$.

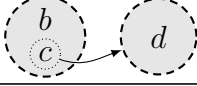
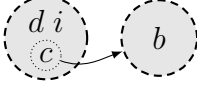
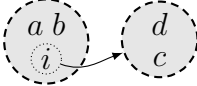
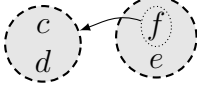
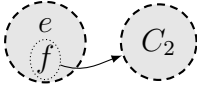
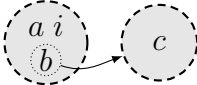
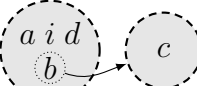



Conditions	Transformation	Results
-		$\alpha_{b,c} = \alpha_{c,d} \stackrel{def}{=} \gamma$
$\forall i \in V \setminus \{a, b, c, d\}$	 	$\alpha_{c,i} = 0$ $\alpha_{b,i} = 0$
if $d \geq 4$ $\forall e, f \in \{1, 2, 3\}$ $\{a, b\} \subset C_3$		$\beta = 0$
if $d \leq 3$ $\forall e, f \in \{1, 2, 3\} \setminus \{d\}$ $C_2 \subset V \setminus \{b, c, d, e, f\}$ $C_3 = \{b, c, d\}$		$\beta = 0$
$\forall i \in V \setminus \{a, b, c, d\}$	    	$\alpha_c + \alpha_{a,b} = \gamma + \alpha_b$ $\alpha_{b,d} = 0$ $\alpha_{a,c} = -\gamma$ $\alpha_{a,b} = \gamma, \alpha_b = \alpha_c$ $\alpha_b = \gamma$

Table 5: Transformations used in Theorem 9.3. Each line presents a step of the proof. The last column corresponds to the result. column.

Proof. Assume that F_P is included in the hyperplane induced by $\alpha^T x = \alpha_0$. Let i and j be two distinct nodes in $V \setminus \{a, b, c, d\}$. Lemma 3.1 can be used with $\{1, i\} \subset C_1$, $\{2, j\} \subset C_2$ and $C_3 = \{b, c\}$ to prove that $\alpha_{i,j} = 0$.

To show that $\alpha_{2,i}$ is equal to zero we apply the same lemma with $\{a, d\} \subset C_1$, $\{b, i\} \subset C_2$. To ensure that the K -partition considered are in F_P , C_3 is equal to c if $d = 1$ and $\{c\}$ is in C_2 otherwise.

We then use Lemma 3.2 with $\{1, b\} \subset C_1$ and $\{2, i\} \subset C_2$ to show that $\alpha_{1,i}$ is equal to zero. This time the validity of the K -partitions is ensured by setting $C_3 = \{c\}$ if d is equal to two and $c \in C_1$ otherwise.

The value of the remaining α coefficients can be obtained through the transformations represented table 6.

□

10. Numerical experiments

In this section we study the strength of our formulation and of the reinforcements with facets of the previous sections. We consider three data sets generated randomly, and we believe the instances to be quite difficult as there are no preexisting classes to detect. Data sets D_1 , D_2 , D_3 all contain 100 instances formed from complete graphs. In D_1 , D_2 and D_3 , the edge weights are respectively in $[0, 500]$, $[-250, 250]$ and $[-500, 0]$.

We first compare the value of the linear relaxation from our formulation and the one from the formulation of Chopra and Rao [1] (also in [14, 10]). The results over the three data sets are displayed in tables 7, 8 and 9. In each table and for each couple (n, K) the value corresponding to formulation (P_2) is the second one. Our formulation gives better relaxation values in all cases except when is equal to 2.

We now only focus on formulation (P_2) . Tables 10 and 11 show the number of instances whose linear relaxation gives an optimal integer solution for data set D_1 and D_2 . No optimal solution is obtained for the instances of D_3 . Data set D_1 has the hardest instances in practice. D_2 instances have weights of both signs, like in [20], and D_3 corresponds to a variant which is considered to be easier (minimizing a cut with K parts and positive weights).

To evaluate the efficiency of a family of inequalities, we use a separation algorithm to add some of them to the formulation and observe the percentage of improvement of the value of the linear relaxation. This necessitates the definition of separation algorithms for each of the family considered.

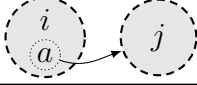
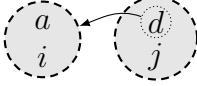
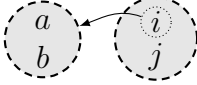
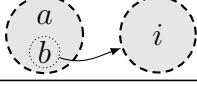
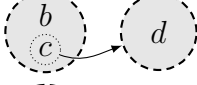
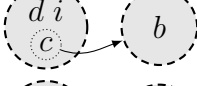
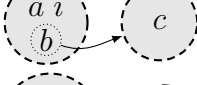
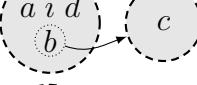
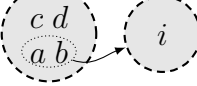
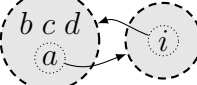
Conditions	Transformation	Results
$\{b, c, d\} \subset C_3$		$\alpha_i = \alpha_j \stackrel{def}{=} \beta$
$\{b, c\} \subset C_3$		$\alpha_{a,d} = -\beta$
$\{c, d\} \subset C_3$		$\alpha_{b,i} = 0$
		$\alpha_{a,b} = -\beta$
-		$\alpha_{b,c} = \alpha_{c,d} \stackrel{def}{=} \gamma$
		$\alpha_{c,i} = 0$
		$\alpha_c - \beta = \gamma$
		$\alpha_{b,d} = 0$
		$\alpha_{a,c} = -\gamma$
		$\gamma = -\beta, \alpha_c = 0$

Table 6: Transformations used in Theorem 9.4. Each line presents a step of the proof. The last column corresponds to the result.

n	K								
	2	3	4	5	6	7	8	9	10
10	1102	114	40	16	7	3	2	1	0
	978	679	462	295	172	88	39	11	0
11	1268	123	45	19	8	4	2	1	0
	1033	737	522	356	227	135	71	33	10
12	1408	140	49	20	9	4	2	1	0
	1124	816	593	422	288	187	112	59	27
13	1529	125	46	20	8	4	2	1	0
	1177	874	649	476	341	231	149	90	47
14	1607	124	46	17	8	4	2	1	0
	1207	904	687	521	387	278	191	123	72
15	1733	125	48	20	8	4	2	1	0
	1275	971	749	578	440	327	234	159	100
16	1883	127	44	19	8	4	2	1	0
	1284	993	784	623	490	378	284	206	142
17	2010	118	43	19	8	4	2	1	0
	1375	1079	858	684	542	426	329	246	177
18	2055	117	41	18	9	4	2	1	0
	1391	1105	893	726	589	474	377	293	221
19	2270	133	49	21	9	4	2	1	1
	1463	1164	941	765	621	503	404	317	243
20	2359	123	43	17	8	4	2	1	0
	1439	1146	936	774	642	530	433	348	273

Table 7: Mean value of the linear relaxation from formulation (P_2) and the formulation proposed by Chopra and Rao [1] over the data set D_1 . For each couple (n, K) the value on the second line corresponds to formulation (P_2).

n	K								
	2	3	4	5	6	7	8	9	10
10	-2246	-2251	-2213	-2146	-2064	-1940	-1754	-1409	0
	-1566	-1620	-1570	-1431	-1236	-990	-709	-385	0
11	-2784	-2778	-2740	-2672	-2591	-2484	-2341	-2136	-1731
	-1861	-1919	-1869	-1746	-1563	-1330	-1062	-752	-407
12	-3374	-3352	-3314	-3245	-3159	-3057	-2931	-2775	-2532
	-2115	-2171	-2149	-2050	-1891	-1679	-1422	-1131	-804
13	-4159	-4140	-4091	-4017	-3933	-3833	-3717	-3577	-3389
	-2557	-2606	-2576	-2480	-2326	-2119	-1867	-1575	-1250
14	-4936	-4901	-4846	-4768	-4674	-4566	-4452	-4318	-4157
	-2938	-2997	-2978	-2891	-2747	-2556	-2318	-2037	-1719
15	-5732	-5693	-5634	-5552	-5467	-5361	-5243	-5115	-4965
	-3332	-3399	-3388	-3314	-3180	-2994	-2759	-2483	-2171
16	-6683	-6641	-6582	-6496	-6410	-6306	-6193	-6067	-5919
	-3803	-3861	-3850	-3779	-3648	-3467	-3242	-2970	-2662
17	-7510	-7484	-7428	-7347	-7257	-7150	-7038	-6919	-6783
	-4250	-4308	-4304	-4240	-4117	-3945	-3726	-3463	-3161
18	-8539	-8510	-8449	-8364	-8272	-8163	-8050	-7925	-7784
	-4788	-4839	-4829	-4768	-4657	-4492	-4277	-4014	-3711
19	-9606	-9559	-9501	-9412	-9319	-9209	-9093	-8960	-8828
	-5300	-5361	-5357	-5306	-5199	-5041	-4839	-4588	-4300
20	-10770	-10725	-10666	-10576	-10483	-10374	-10259	-10135	-10000
	-5936	-5991	-5979	-5915	-5797	-5628	-5410	-5153	-4860

Table 8: Mean value of the linear relaxation from formulation (P_2) and the formulation proposed by Chopra and Rao [1] over the data set D_2 . For each couple (n, K) the value on the second line corresponds to formulation (P_2).

n	K								
	2	3	4	5	6	7	8	9	10
10	-10929	-10438	-9935	-9294	-8622	-7848	-6925	-5549	0
	-9959	-8724	-7488	-6253	-5017	-3780	-2540	-1293	0
11	-13490	-13001	-12498	-11865	-11191	-10453	-9589	-8557	-6986
	-12375	-11009	-9643	-8277	-6911	-5545	-4174	-2801	-1422
12	-16107	-15611	-15111	-14477	-13802	-13068	-12246	-11317	-10185
	-14876	-13397	-11918	-10439	-8960	-7480	-5999	-4517	-3028
13	-19402	-18912	-18414	-17773	-17102	-16372	-15574	-14680	-13651
	-18019	-16390	-14761	-13131	-11502	-9872	-8243	-6611	-4977
14	-22638	-22135	-21626	-20988	-20311	-19567	-18784	-17912	-16963
	-21141	-19390	-17639	-15888	-14137	-12385	-10634	-8883	-7130
15	-26015	-25517	-25003	-24366	-23716	-22983	-22206	-21357	-20428
	-24402	-22537	-20673	-18808	-16943	-15078	-13214	-11347	-9481
16	-29871	-29372	-28864	-28226	-27573	-26841	-26078	-25250	-24336
	-28121	-26122	-24124	-22125	-20126	-18128	-16129	-14130	-12132
17	-33803	-33308	-32805	-32174	-31515	-30786	-30034	-29226	-28346
	-31935	-29818	-27702	-25585	-23469	-21352	-19236	-17118	-15001
18	-38126	-37624	-37113	-36473	-35816	-35081	-34329	-33520	-32650
	-36139	-33893	-31648	-29402	-27157	-24911	-22666	-20420	-18174
19	-42640	-42143	-41643	-41001	-40352	-39621	-38871	-38047	-37207
	-40515	-38143	-35771	-33399	-31027	-28655	-26283	-23911	-21539
20	-47565	-47065	-46560	-45916	-45262	-44534	-43788	-42981	-42148
	-45309	-42801	-40294	-37786	-35279	-32771	-30263	-27755	-25247

Table 9: Mean value of the linear relaxation from formulation (P_2) and the formulation proposed by Chopra and Rao [1] over the data set D_3 . For each couple (n, K) the value on the second line corresponds to formulation (P_2).

n	K																		
	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	
10	0	0	0	12	47	75	95	100											
11	0	0	0	0	15	50	77	94	100										
12	0	0	0	0	8	29	70	91	99	100									
13	0	0	0	0	0	8	37	60	87	99	100								
14	0	0	0	0	0	0	20	45	72	91	98	100							
15	0	0	0	0	0	0	6	28	61	84	94	99	100						
16	0	0	0	0	0	0	3	13	31	66	80	91	98	100					
17	0	0	0	0	0	0	0	3	18	46	69	84	90	96	100				
18	0	0	0	0	0	0	0	0	6	19	51	70	91	97	99	100			
19	0	0	0	0	0	0	0	0	1	13	34	56	76	92	96	99	100		
20	0	0	0	0	0	0	0	0	0	4	12	37	60	78	94	99	100	100	

Table 10: Number of instances of D_1 whose linear relaxation gives an optimal solution.

The values of n considered in our experiments are low enough to allow an exhaustive enumeration of all the valid paw inequalities. Separating the 2-partition inequalities is NP-hard [8] and we are not able to enumerate them all. Instead we use a heuristic inspired from the well-known Kernighan-Lin algorithm [21]. A similar procedure is used for the separation of the general clique inequalities.

Separating the 2-chorded cycle inequalities is a bit more technical. In [22], Müller adapted an approach, introduced by Barahona and Mahjoub [5], to separate in polynomial time odd closed walk inequalities in directed graphs. Müller showed that the same algorithm can be applied to undirected graphs to allow the separation of a class of inequalities which includes the 2-chorded cycle inequalities. We adapted this approach to separate 2-chorded cycle inequalities from cycles which may contain repetitions.

n	K																		
	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	
10	20	53	31	4	3	0	0	0											
11	16	37	16	4	1	1	0	0	0										
12	8	21	16	4	0	0	0	0	0	0									
13	12	17	11	2	1	0	0	0	0	0	0								
14	6	14	9	3	0	0	0	0	0	0	0	0							
15	2	10	9	3	0	0	0	0	0	0	0	0	0						
16	2	5	5	1	0	0	0	0	0	0	0	0	0	0					
17	2	10	4	2	0	0	0	0	0	0	0	0	0	0	0				
18	2	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0			
19	1	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0		
20	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	

Table 11: Number of instances of D_2 whose linear relaxation gives an optimal solution.

We define a graph $H = (V_H, A_H)$ such that for each edge $ij \in E$, A_H contains (see example Figure 27):

- eight vertices: $u_1^{ij}, u_2^{ij}, v_1^{ij}, v_2^{ij}, u_1^{ji}, u_2^{ji}, v_1^{ji}$ and v_2^{ji} ;
- four arcs: $(u_1^{ij}, u_2^{ij}), (v_1^{ij}, v_2^{ij}), (u_1^{ji}, u_2^{ji}), (v_1^{ji}, v_2^{ji})$ of weight x_{ij} .

Moreover, to each pair of edges $ij, ik \in E$ with a common endnode, we associate four additional arcs in A_H : $(u_2^{ji}, v_1^{ik}), (v_2^{ji}, u_1^{ik}), (u_2^{ki}, v_1^{ij})$ and (v_2^{ki}, u_1^{ij}) of weight $-x_{jk} - \frac{1}{2}$.

Let $C = \{c_1, \dots, c_{2p+1}\}$ be an odd cycle of G . By construction, C induces a walk in H from $u_1^{c_1, c_2}$ to $v_1^{c_1, c_2}$ (see example Figure 28) of weight

$$\begin{aligned}
& x_{c_1, c_2} - \frac{1}{2} - x_{c_1, c_3} + \dots + x_{c_{2p+1}, c_1} - \frac{1}{2} - x_{c_{2p+1}, c_2} \\
& = x(E(C)) - x(E(\overline{C})) - \frac{2p+1}{2} \\
& = x(E(C)) - x(E(\overline{C})) - \lfloor \frac{|C|}{2} \rfloor - \frac{1}{2}.
\end{aligned}$$

Thus, there exists a cycle C which violates inequality (21) if and only if there exists a path from $u_1^{c_1, c_2}$ to $v_1^{c_1, c_2}$ in H whose length is greater than $-\frac{1}{2}$.

Müller's approach for undirected graphs only considers four vertices per edge ($u_1^{ij}, u_2^{ij}, v_1^{ij}$ and v_2^{ij}). As a consequence, a path in H between u_1^{ij} and v_1^{ij} corresponds to a sequence of edges in G such that each edge has a common endnode with its neighbors. Such a sequence may not be a cycle (*e.g.*: $\{ij, ik, il\}$). Four additional vertices per edge enable to give an orientation to the edge in the obtained sequence and thus ensure that it is a cycle (possibly with vertex repetitions).

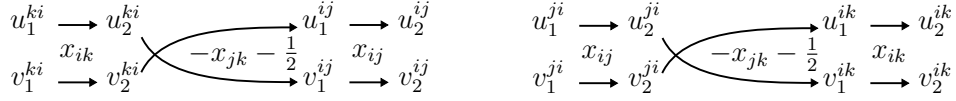


Figure 27: Vertices and arcs of H associated to edges (ij) and (ik) in E .

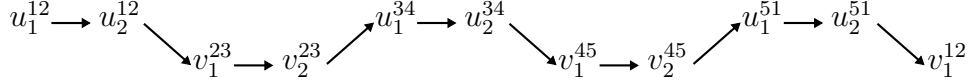


Figure 28: Path in H which corresponds to the cycle $C = \{1, 2, 3, 4, 5\}$ in G .

After creating H , we obtain for all $ij \in E$ the shortest path between u_1^{ij} and v_1^{ij} thanks to Floyd-Warshall shortest path algorithm [23]. And deduce the corresponding cycle in G and its associated 2-chorded cycle inequality. Eventually, the violated inequalities are added to the problem and the root relaxation is updated. This process is repeated until no more violated inequality is found.

For the instances in D_1 – which are the most difficult ones – no violated paw inequality has been found. This family of inequalities does not seem to be efficient in this case. Table 12 shows the results obtained when adding 2-chorded cycle inequalities. The mean percentage of improvement is low (lower than 5%). These inequalities are less likely to be efficient for this type of instances. Moreover, the improvement decreases when K gets closer to n . This is true also for the other classes of inequalities, and it can be explained by the fact that the number of solutions solved to optimality increases in this part of the tables.

Table 13 and 14 present respectively the results obtained with the 2-partition and the general clique inequalities. The improvement is significantly higher than for the 2-chorded cycle inequalities. The general clique inequalities lead to a spectacular improvement of the solution of the linear relaxation for the lowest values of K .

n	K								
	2	3	4	5	6	7	8	9	10
10	1,2	1,5	0,9	0,7	0,1	0,1	0,0	0,0	
11	0,9	1,1	0,9	0,6	0,2	0,0	0,0	0,0	0,0
12	1,7	1,9	1,7	1,4	1,0	0,6	0,2	0,0	0,0
13	1,8	2,0	1,9	1,5	0,6	0,3	0,0	0,0	0,0
14	2,2	2,5	2,5	2,1	1,3	0,7	0,4	0,1	0,0
15	2,1	2,9	3,0	2,5	1,9	1,1	0,5	0,2	0,0
16	2,5	2,8	2,7	2,1	1,7	1,4	0,7	0,4	0,1
17	2,9	3,2	3,0	2,7	2,3	1,7	1,4	0,7	0,3
18	2,9	3,4	3,3	2,9	2,5	1,9	1,3	0,8	0,4
19	3,2	3,6	3,9	3,7	3,2	2,6	2,0	1,2	0,7
20	4,1	4,7	4,8	4,6	4,1	3,4	3,0	2,6	1,6

Table 12: Mean gain in percentage over the instances of D_1 when adding 2-chorded cycle inequalities.

In the case of the instances of D_2 , the 2-chorded cycle inequality still provide low gains as represented in table 15. For a fix value of n we observe that the mean improvement does not vary depending on K except for the top right corner of the table. These couples (n, K) correspond to the instances which may directly be solved optimally by the relaxation (as represented table 11).

The results for the 2-partition inequalities, represented in Figure 16, are similar. Although lower than for D_1 , the mean gains are twice as high as those given by the 2-chorded cycle inequalities.

Table 17 shows that the gain is significantly lower for the general clique inequalities. These inequalities require the presence of a minimum number of edges from a clique $E(Z)$ in the K -partition. Since the we minimize the weight of the edges in the K sets, an instance of D_1 will tend to have less edges in the sets than an instance of D_2 . Nevertheless, these inequalities can still be useful for the small values of K .

As for the paw inequalities (table 18), they seem to complement the general clique inequalities well: the gain increases with K .

n	K								
	2	3	4	5	6	7	8	9	10
10	7,5	9,3	8,1	7,1	3,5	1,6	0,5	0,0	
11	9,6	10,9	10,6	8,2	5,3	2,9	1,6	0,8	0,0
12	11,4	13,2	12,6	10,6	7,7	4,2	2,8	1,1	0,1
13	12,8	15,3	15,6	13,5	9,5	6,2	3,4	2,1	0,5
14	14,9	17,0	17,8	15,7	12,8	9,4	5,7	3,1	1,8
15	16,9	19,6	20,5	19,3	17,3	14,3	10,6	6,1	3,5
16	17,8	19,9	20,5	19,8	17,9	14,9	11,3	8,2	5,0
17	19,1	21,5	22,1	21,3	20,2	17,3	14,0	10,6	7,3
18	21,1	23,3	23,8	23,2	21,3	18,7	15,6	12,1	8,4
19	22,5	25,0	26,0	26,2	25,7	23,5	20,1	16,5	12,1
20	25,1	28,6	30,1	29,6	28,4	26,1	22,8	18,9	14,4

Table 13: Mean gain in percentage over the instances of D_1 when adding 2-partition inequalities.

When instances with only negative weights are considered, we fail at finding any violated inequality except for the paw inequalities (19). These instances are however easier to solve in practice.

Conclusion

We have introduced a new formulation for the K-partitioning problem. Thanks to the addition of representative variables, we are able to break the symmetry in the edge variable formulation. The resulting formulation shows to be stronger than the formulation with node-cluster variables and edge variables used by several authors ([1, 14, 10]) when K is greater than 2, at least on complete graphs. We have proved in this paper facet-defining results for several classical families of inequalities, and for a new family of inequalities that seems to be useful when there are negative edges.

The computing time for the 20-node instances is only of a few minutes using CPLEX 12.5 on a desktop computer. To actually solve problems to optimality for higher values of n will need to find a compromise between the separation and the solving of the linear programs at the nodes of a branch

n	K								
	2	3	4	5	6	7	8	9	10
10	298,0	153,6	73,7	32,2	15,9	4,3	0,4	0,0	
11	368,7	197,8	104,3	49,6	17,8	7,5	3,8	0,8	0,0
12	434,0	239,2	135,5	73,0	31,2	12,4	5,4	2,3	0,6
13	505,7	288,4	171,5	98,5	49,2	19,9	9,1	5,1	2,0
14	593,6	343,6	206,9	119,2	68,2	33,2	14,7	6,3	3,4
15	673,2	395,4	244,7	152,0	95,5	54,2	25,5	13,6	6,1
16	782,9	465,0	294,0	181,8	112,1	68,1	35,4	17,6	9,8
17	855,6	515,5	328,1	214,8	137,8	84,9	48,2	23,7	13,4
18	964,1	582,2	376,3	247,7	159,8	102,0	63,6	36,9	19,6
19	1041,6	637,4	421,9	289,1	199,5	134,0	88,2	52,4	26,8
20	1186,6	739,3	491,1	336,3	228,9	153,1	103,1	63,5	35,9

Table 14: Mean gain in percentage over the instances of D_1 when adding general clique inequalities.

and bound procedure. Still the results of this work are promising and show the interest of the polyhedral approach.

Further work will concentrate on improving the separation procedures and developing a branch and cut framework for the application that motivated this study [9].

11. References

- [1] S. Chopra, M. Rao, The partition problem, *Mathematical Programming* 59 (1) (1993) 87–115.
- [2] C. Bichot, P. Siarry, *Graph Partitioning*, ISTE, John Wiley & Sons, London, 2013.
- [3] C. Ferreira, A. Martin, C. De Souza, R. Weismantel, L. Wolsey, The node capacitated graph partitioning problem: a computational study, *Mathematical Programming* 81 (2) (1998) 229–256.

n	K								
	2	3	4	5	6	7	8	9	10
10	0,4	0,4	0,4	0,6	0,8	0,8	0,4	0,0	
11	0,4	0,4	0,4	0,6	1,0	1,5	1,2	0,5	0,0
12	0,9	0,7	0,7	0,9	1,3	2,1	2,5	1,7	0,7
13	1,0	1,0	1,0	1,1	1,3	1,9	2,7	2,8	1,7
14	1,2	1,2	1,2	1,3	1,5	1,9	2,9	3,5	2,8
15	2,1	2,1	2,2	2,3	2,5	2,7	3,5	4,2	4,4
16	2,6	2,7	2,7	2,7	2,8	3,1	3,6	4,7	5,2
17	3,1	3,2	3,2	3,3	3,3	3,4	3,7	4,5	5,5
18	4,0	4,0	4,0	4,0	4,1	4,2	4,3	4,8	5,9
19	4,5	4,8	4,9	5,0	5,0	5,0	5,0	5,2	5,9
20	5,5	5,8	5,8	5,8	5,8	5,8	5,9	6,0	6,4

Table 15: Mean gain in percentage over the instances of D_2 when adding 2-chorded cycle inequalities.

- [4] E. Johnson, A. Mehrotra, G. Nemhauser, Min-cut clustering, *Mathematical Programming* 62 (1) (1993) 133–151.
- [5] F. Barahona, A. R. Mahjoub, On the cut polytope, *Mathematical programming* 36 (2) (1986) 157–173.
- [6] M. R. Garey, D. S. Johnson, L. Stockmeyer, Some simplified np-complete graph problems, *Theoretical computer science* 1 (3) (1976) 237–267.
- [7] M. Grötschel, Y. Wakabayashi, Facets of the clique partitioning polytope, *Mathematical Programming* 47 (1) (1990) 367–387.
- [8] M. Oosten, J. Rutten, F. Spiessma, The clique partitioning problem: facets and patching facets, *Networks* 38 (4) (2001) 209–226.
- [9] Z. Ales, G. D. Duplessis, O. Serban, A. Pauchet, A methodology to design human-like embodied conversational agents based on dialogue analysis, in: *Proceedings of the 11th international conference on Au-*

n	K								
	2	3	4	5	6	7	8	9	10
10	1,8	1,4	1,2	2,2	3,8	6,4	10,1	13,5	
11	2,3	1,9	1,8	2,6	4,0	6,9	9,1	11,7	13,2
12	3,0	2,3	2,2	2,9	4,4	6,6	9,5	11,8	13,2
13	3,7	3,4	3,5	3,9	4,8	6,6	9,6	12,5	14,4
14	4,3	4,0	3,9	4,2	5,0	6,5	8,5	11,1	13,1
15	5,3	5,0	5,1	5,3	5,9	6,8	8,8	11,4	14,0
16	5,6	5,5	5,5	5,6	6,1	7,0	8,3	11,0	13,1
17	7,3	7,1	7,3	7,3	7,6	8,1	8,8	10,5	13,1
18	8,0	7,7	7,9	7,9	8,2	8,3	9,0	10,3	12,5
19	9,2	9,5	9,6	9,6	9,9	9,8	10,3	11,1	12,5
20	10,5	10,6	10,6	10,6	10,8	10,7	10,9	11,7	12,7

Table 16: Mean gain in percentage over the instances of D_2 when adding 2-partition inequalities.

onomous Agents and Multiagent Systems (AAMAS) at Human-Agent Interaction Design and Models (HAIDM), 2012, pp. 34–49.

- [10] N. Fan, P. M. Pardalos, Linear and quadratic programming approaches for the general graph partitioning problem, *Journal of Global Optimization* 48 (1) (2010) 57–71.
- [11] W. W. Hager, D. T. Phan, H. Zhang, An exact algorithm for graph partitioning, *Mathematical Programming* 137 (1-2) (2013) 531–556.
- [12] P. Bonami, V. H. Nguyen, M. Klein, M. Minoux, On the solution of a graph partitioning problem under capacity constraints, in: *Proceedings of the 2nd International Symposium on Combinatorial Optimization (ISCO)*, LNCS, Vol. 7422, Springer, 2012, pp. 285–296.
- [13] C. Ferreira, A. Martin, C. de SOUZA, R. Weismantel, L. Wolsey, Formulations and valid inequalities for the node capacitated graph partitioning problem, *Mathematical Programming* 74 (3) (1996) 247–266.

n	K								
	2	3	4	5	6	7	8	9	10
10	16,7	2,6	0,2	0,0	0,0	0,0	0,0	0,0	
11	16,2	2,7	0,4	0,0	0,0	0,0	0,0	0,0	0,0
12	17,5	3,2	0,5	0,1	0,0	0,0	0,0	0,0	0,0
13	17,2	4,0	0,8	0,1	0,0	0,0	0,0	0,0	0,0
14	17,3	4,3	0,8	0,1	0,0	0,0	0,0	0,0	0,0
15	19,5	5,7	1,2	0,1	0,0	0,0	0,0	0,0	0,0
16	20,2	6,1	1,6	0,2	0,0	0,0	0,0	0,0	0,0
17	22,5	7,7	2,1	0,3	0,0	0,0	0,0	0,0	0,0
18	23,6	8,1	2,5	0,4	0,0	0,0	0,0	0,0	0,0
19	24,8	9,2	2,9	0,5	0,0	0,0	0,0	0,0	0,0
20	24,9	9,8	3,2	0,5	0,0	0,0	0,0	0,0	0,0

Table 17: Mean gain in percentage over the instances of D_2 when adding general clique inequalities.

- [14] V. Kaibel, M. Peinhardt, M. E. Pfetsch, Orbitopal fixing, Discrete Optimization 8 (4) (2011) 595–610.
- [15] M. Campêlo, V. A. Campos, R. C. Corrêa, On the asymmetric representatives formulation for the vertex coloring problem, Discrete Applied Mathematics 156 (7) (2008) 1097–1111.
- [16] M. Labbé, F. Özsoy, Size-constrained graph partitioning polytopes, Discrete Mathematics 310 (24) (2010) 3473–3493.
- [17] G. L. Nemhauser, L. A. Wolsey, Integer and Combinatorial Optimization, Wiley-Interscience, New York, NY, USA, 1988.
- [18] M. Deza, M. Grötschel, M. Laurent, Clique-web facets for multicut polytopes, Mathematics of Operations Research (1992) 981–1000.
- [19] X. Ji, J. Mitchell, The clique partitioning problem with minimum size requirement, Tech. rep., Technical report, Rensselaer Polytechnic Institute (2005).

n	K								
	2	3	4	5	6	7	8	9	10
10	0,0	0,1	1,4	5,2	11,0	18,5	26,5	35,3	
11	0,0	0,1	1,2	4,4	9,7	16,7	23,8	30,4	37,3
12	0,0	0,0	0,6	2,9	7,7	14,4	21,7	27,8	33,4
13	0,0	0,1	0,6	2,3	6,1	12,2	19,3	26,3	32,2
14	0,0	0,0	0,3	1,5	4,5	9,9	16,5	22,8	28,5
15	0,0	0,0	0,1	0,8	3,0	7,6	13,9	20,6	26,7
16	0,0	0,0	0,1	0,6	2,2	6,1	11,8	18,5	24,7
17	0,0	0,0	0,1	0,4	1,4	4,1	9,0	15,3	21,8
18	0,0	0,0	0,1	0,2	0,8	2,8	7,2	13,1	19,6
19	0,0	0,0	0,0	0,1	0,4	1,6	5,0	10,1	16,2
20	0,0	0,0	0,0	0,1	0,2	1,0	3,8	8,6	14,6

Table 18: Mean gain in percentage over the instances of D_2 when adding paw inequalities.

- [20] M. Grötschel, Y. Wakabayashi, A cutting plane algorithm for a clustering problem, *Mathematical Programming* 45 (1) (1989) 59–96.
- [21] B. W. Kernighan, S. Lin, An efficient heuristic procedure for partitioning graphs, *Bell system technical journal* 49 (2) (1970) 291–307.
- [22] R. Müller, On the partial order polytope of a digraph, *Mathematical Programming* 73 (1) (1996) 31–49.
- [23] R. K. Ahuja, T. L. Magnanti, J. B. Orlin, *Network Flows: Theory, Algorithms, and Applications*, Prentice Hall, Englewood Cliffs, NJ, 1993.

n	K								
	2	3	4	5	6	7	8	9	10
10	0,7	1,5	2,5	3,8	5,2	9,6	20,3	32,0	
11	0,7	1,4	2,4	3,6	5,0	6,5	14,2	24,3	33,3
12	0,6	1,4	2,2	3,3	4,6	6,1	9,9	18,2	27,2
13	0,6	1,4	2,3	3,3	4,6	6,0	7,6	13,6	21,5
14	0,6	1,3	2,1	3,1	4,2	5,5	7,0	10,2	16,6
15	0,6	1,3	2,1	3,0	4,1	5,3	6,8	8,4	13,2
16	0,6	1,3	2,0	2,9	3,9	5,1	6,4	7,9	10,5
17	0,6	1,3	2,0	2,9	3,8	4,9	6,2	7,7	9,2
18	0,6	1,2	1,9	2,7	3,6	4,7	5,8	7,1	8,6
19	0,6	1,2	1,9	2,7	3,5	4,5	5,6	6,8	8,2
20	0,6	1,2	1,8	2,6	3,4	4,3	5,3	6,5	7,7

Table 19: Mean gain in percentage over the instances of D_3 when adding paw inequalities.